

Four-loop pressure of massless $O(N)$ scalar field theory

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ABSTRACT: Inspired by the corresponding problem in QCD, we determine the pressure of massless $O(N)$ scalar field theory up to order g^6 in the weak-coupling expansion, where g^2 denotes the quartic coupling constant. This necessitates the computation of all 4-loop vacuum graphs at a finite temperature: by making use of methods developed by Arnold and Zhai at 3-loop level, we demonstrate that this task is manageable at least if one restricts to computing the logarithmic terms analytically, while handling the “constant” 4-loop contributions numerically. We also inspect the numerical convergence of the weak-coupling expansion after the inclusion of the new terms. Finally, we point out that while the present computation introduces strategies that should be helpful for the full 4-loop computation on the QCD-side, it also highlights the need to develop novel computational techniques, in order to be able to complete this formidable task in a systematic fashion.

KEYWORDS: NLO Computations, Thermal Field Theory.

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1. Introduction

Motivated for instance by hydrodynamic studies of heavy ion collision experiments, and dark matter relic density computations in cosmology, a lot of theoretical work has been devoted to the perturbative determination of the pressure of hot QCD in recent years. As a result of 3-loop and 4-loop computations, corrections to the non-interacting Stefan-Boltzmann law have been determined up to relative orders $\mathcal{O}(g^4)$ [1], $\mathcal{O}(g^5)$ [2, 3], and

$\mathcal{O}(g^6 \ln(1/g))$ [4], where g denotes the renormalized strong coupling constant. The first presently unknown order, $\mathcal{O}(g^6)$, contains a non-perturbative coefficient [5, 6], but that can also be estimated numerically [7, 8]. All orders of g are available in the formal limit of large N_f [9], where N_f counts the number of massless quark flavours. Similar results have also been obtained for the case of non-zero quark chemical potentials at finite [10] and large [11] N_f . Moreover, first steps towards the inclusion of finite quark masses, important for phenomenological applications, have been taken [12]. Finally, coefficients up to the order $\mathcal{O}(g^5)$ are available even for the standard electroweak theory, at temperatures higher than the electroweak scale [13].

Conceptually, it would be quite desirable to extend these results up to the full order $\mathcal{O}(g^6)$. The reason is that this is the first order where at least the leading contributions from the various momentum scales relevant for hot QCD, $k \sim 2\pi T, gT, g^2T$, have been fully accounted for. As mentioned, the non-perturbative input needed to describe the effect of the softest momenta $k \sim g^2T$ is already available [7, 8]. Moreover, several perturbative contributions of $\mathcal{O}(g^6)$ are known: in the notation of ref. [4], β_M [14] (which accounts for the scales $k \sim gT$), β_{E2} [15], β_{E3} [16], as well as β_{E4} and β_{E5} [17] have been computed. Nevertheless, a single coefficient, β_{E1} , coming from the hard scales $k \sim 2\pi T$, remains undetermined.

The recipe to compute β_{E1} is in principle simple: it is defined to be the “naive” (i.e. unresummed) 4-loop contribution to the pressure, computed by regulating all divergences via dimensional regularization in $d = 4 - 2\epsilon$ dimensions. This computation contains infrared (IR) divergences, which manifest themselves as an uncancelled $1/\epsilon$ -divergence in the final (renormalized) result. This IR pole gets only cancelled once the contributions of the soft modes are properly resummed, a step that has already been completed, and produces another $1/\epsilon$ -pole [4], with the opposite sign.

Unfortunately, the practical implementation of this computation is far from trivial. In fact, to show that it is feasible at all, it is the purpose of the present paper to demonstrate that the 4-loop order can at least be reached in models somewhat simpler than QCD.

The model that we consider is scalar field theory, with a global $O(N)$ symmetry, i.e. the “ $\lambda\phi^4$ ”-theory. To keep the analogy with QCD in mind, we follow here the frequent convention of denoting $\lambda \equiv g^2$. The pressure of this theory has been computed to high orders in g in parallel with that for QCD: the orders $\mathcal{O}(g^4)$ [18, 1], $\mathcal{O}(g^5)$ [19, 20], and $\mathcal{O}(g^6 \ln(1/g))$ [20] have been reached. The order $\mathcal{O}(g^6)$ involves a coefficient analogous to β_{E1} , which again contains an uncancelled $1/\epsilon$ -pole. It is the goal of the present paper to compute this β_{E1} (as well as all other relevant coefficients), showing that the pole cancels, and thus to determine the full pressure up to $\mathcal{O}(g^6)$, at any finite N .

It is perhaps worth stressing that even though the technical challenges addressed in this paper have a direct counterpart in QCD, there is of course the conceptual difference that in our model the order $\mathcal{O}(g^6)$ contains no non-perturbative coefficients, since the scale $k \sim g^2T$ does not exist in scalar field theory. Therefore no lattice studies of the type in refs. [7, 8] need to be invoked here.

The plan of this paper is the following. We start by carrying out the naive 4-loop computation of the pressure in section 2, leading to the determination of the coefficient

β_{E1} . In section 3, we elaborate on how the naive computation can be re-interpreted and incorporated in a proper setting such that all divergences cancel. This leads to our final finite result. We discuss various formal and numerical aspects of the result in section 4, and conclude in section 5.

2. Naive 4-loop computation

Our starting point is the bare theory

$$\mathcal{L}_B = \frac{1}{2} \sum_{\mu=1}^{4-2\epsilon} \sum_{i=1}^N \partial_\mu \phi_i \partial_\mu \phi_i + \frac{1}{4!} g_B^2 \Lambda^{2\epsilon} \left(\sum_{i=1}^N \phi_i \phi_i \right)^2, \quad (2.1)$$

where Λ is the scale parameter introduced in connection with dimensional regularization. We work in Euclidean metric throughout. The fields ϕ_i have the dimension $[\text{GeV}]^{1-\epsilon}$, while the bare coupling g_B^2 is dimensionless. The theory can be renormalized in the $\overline{\text{MS}}$ scheme by setting

$$g^2 = g^2 + \frac{g^4}{(4\pi)^2} \frac{\beta_1}{\epsilon} + \frac{g^6}{(4\pi)^4} \left(\frac{\beta_1^2}{\epsilon^2} + \frac{\beta_2}{2\epsilon} \right) + \mathcal{O}(g^8), \quad (2.2)$$

where (see, e.g., ref. [21])

$$\beta_1 = \frac{N+8}{6}, \quad \beta_2 = -\frac{3N+14}{6}. \quad (2.3)$$

The observable that we are interested in, is the pressure, or minus the free energy density, of this theory, as function of the temperature T and the renormalized coupling constant g . Formally, the pressure is given by the path integral

$$p(T) \equiv \lim_{V \rightarrow \infty} \frac{T}{V} \ln \int \mathcal{D}\phi_i \exp(-S_B), \quad (2.4)$$

where V is the volume, $V = \int d^{3-2\epsilon} \mathbf{x}$, and S_B is the bare action, $S_B = \int_0^\beta d\tau \int d^{3-2\epsilon} \mathbf{x} \mathcal{L}_B$. The temperature T is given by $T = 1/\beta$, and the path integral is taken with the usual periodic boundary conditions over the τ -direction. Note that in the presence of dimensional regularization, the pressure thus defined has the dimension $[\text{GeV}]^{4-2\epsilon}$.

Going into momentum space, we define the usual (sum-)integrals as

$$\int_p \equiv \Lambda^{2\epsilon} \int \frac{d^{3-2\epsilon} \mathbf{p}}{(2\pi)^{3-2\epsilon}} = \left(\frac{e^{\gamma_E} \bar{\Lambda}^2}{4\pi} \right)^\epsilon \int \frac{d^{3-2\epsilon} \mathbf{p}}{(2\pi)^{3-2\epsilon}}, \quad (2.5)$$

$$\not\int_P \equiv T \sum_{p_0} \int_p, \quad (2.6)$$

where p_0 stands for bosonic Matsubara momenta, $p_0 = 2\pi nT$, $n \in \mathbb{Z}$. Furthermore, Λ is the $\overline{\text{MS}}$ scale parameter, while $\bar{\Lambda}$ is the $\overline{\text{MS}}$ one; these are related through $\bar{\Lambda}^2 \equiv 4\pi \Lambda^2 \exp(-\gamma_E)$.

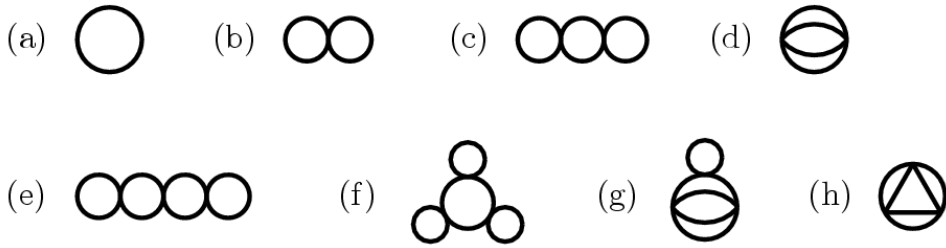


Figure 1: The 1-, 2-, 3- and 4-loop full theory diagrams contributing to the pressure.

We then define

$$\mathcal{I}_n^m \equiv \int_P \frac{(p_0)^m}{(P^2)^n} , \quad (2.7)$$

$$\Pi(P) \equiv \int_Q \frac{1}{Q^2(Q-P)^2} , \quad (2.8)$$

$$\bar{\Pi}(P) \equiv \int_Q \frac{1}{Q^4(Q-P)^2} . \quad (2.9)$$

Up to 4-loop order, the strict loop expansion for the pressure of our theory contains the graphs (a)–(h) in figure 1, which we denote by the symbols I_a – I_h . Denoting furthermore $\mathcal{I}_n \equiv \mathcal{I}_n^0$, and employing the sum-integrals defined in eqs. (2.7)–(2.9), we can immediately write down their expressions in the forms:

$$I_a = N \frac{\pi^2}{90} T^4 [1 + \mathcal{O}(\epsilon)] , \quad (2.10)$$

$$I_b = -\frac{N(N+2)}{24} g_B^2 (\mathcal{I}_1)^2 , \quad (2.11)$$

$$I_c = \frac{N(N+2)^2}{144} g_B^4 (\mathcal{I}_1)^2 \mathcal{I}_2 , \quad (2.12)$$

$$I_d = \frac{N(N+2)}{144} g_B^4 \int_P [\Pi(P)]^2 , \quad (2.13)$$

$$I_e = -\frac{N(N+2)^3}{864} g_B^6 (\mathcal{I}_1)^2 (\mathcal{I}_2)^2 , \quad (2.14)$$

$$I_f = -\frac{N(N+2)^3}{1296} g_B^6 (\mathcal{I}_1)^3 \mathcal{I}_3 , \quad (2.15)$$

$$I_g = -\frac{N(N+2)^2}{216} g_B^6 \mathcal{I}_1 \int_P \Pi(P) \bar{\Pi}(P) , \quad (2.16)$$

$$I_h = -\frac{N(N+2)(N+8)}{1296} g_B^6 \int_P [\Pi(P)]^3 . \quad (2.17)$$

Taking now into account the renormalization of the coupling constant up to 2-loop order, we must replace the bare coupling g_B^2 by the renormalized one, g^2 , through eq. (2.2). Recalling that a weak-coupling expansion (i.e., an expansion in g^2) of the path-integral in eq. (2.4) does not coincide with the loop expansion at finite temperatures, because of well-known

IR divergences in multiloop graphs, we will denote the result for the sum of these graphs in the following by $p_E(T)$, in contrast to the physical pressure denoted by $p(T)$; for the latter we assume that a consistent evaluation has been carried out to a certain order in g^2 , irrespective of how many loop orders this takes. Summing together the graphs shown leads then to the following expression for $p_E(T)$:

$$\begin{aligned} \Lambda^{2\epsilon} p_E &= N \frac{\pi^2}{90} T^4 [1 + \mathcal{O}(\epsilon)] - g^2 \times \frac{N(N+2)}{24} (\mathcal{I}_1)^2 + \\ &+ g^4 \times \frac{N(N+2)}{144} \left\{ (\mathcal{I}_1)^2 \left[(N+2)\mathcal{I}_2 - \frac{N+8}{(4\pi)^2\epsilon} \right] + \not\int_P [\Pi(P)]^2 \right\} - \\ &- g^6 \times \frac{N(N+2)}{1296} \left\{ (N+2)^2 (\mathcal{I}_1)^3 \mathcal{I}_3 + \frac{3}{2} (\mathcal{I}_1)^2 \left[(N+2)^2 (\mathcal{I}_2)^2 - \right. \right. \\ &- \left. \frac{2(N+2)(N+8)}{(4\pi)^2\epsilon} \mathcal{I}_2 + \frac{1}{(4\pi)^4} \left(\frac{(N+8)^2}{\epsilon^2} - \frac{3(3N+14)}{\epsilon} \right) \right] + \\ &+ 6(N+2)\mathcal{I}_1 \not\int_P \Pi(P)\bar{\Pi}(P) + (N+8) \not\int_P [\Pi(P)]^3 - \frac{3(N+8)}{(4\pi)^2\epsilon} \not\int_P [\Pi(P)]^2 \left. \right\}. \end{aligned} \tag{2.18}$$

From ref. [1], we can immediately read off the results for most of the sum-integrals above, utilizing the formulae

$$\mathcal{I}_n^m = 2^{m-2n+1} \pi^{m-2n+3/2} T^{m-2n+4} \left(\frac{\Lambda^2}{\pi T^2} \right)^\epsilon \frac{\Gamma(n-3/2+\epsilon)}{\Gamma(n)} \zeta(2n-m-3+2\epsilon), \tag{2.19}$$

$$\not\int_P [\Pi(P)]^2 = \frac{1}{(4\pi)^2} \left(\frac{T^2}{12} \right)^2 \left\{ \frac{6}{\epsilon} + 36 \ln \frac{\bar{\Lambda}}{4\pi T} + 48 \frac{\zeta'(-1)}{\zeta(-1)} - 12 \frac{\zeta'(-3)}{\zeta(-3)} + \frac{182}{5} + \mathcal{O}(\epsilon) \right\}. \tag{2.20}$$

This leaves us with the task of evaluating the new 3- and 4-loop sum-integrals

$$S_1 \equiv \not\int_P \Pi(P)\bar{\Pi}(P), \tag{2.21}$$

$$S_2 \equiv \not\int_P \left\{ [\Pi(P)]^3 - \frac{3}{(4\pi)^2\epsilon} [\Pi(P)]^2 \right\}, \tag{2.22}$$

where we have grouped together two terms in S_2 for computational convenience.

The sum-integrals in eqs. (2.21), (2.22) can be evaluated through a very tedious if in principle straightforward application of the procedures and techniques that were pioneered by Arnold and Zhai in ref. [1]. A detailed explanation of the steps that we have taken can be found in appendix A. Here we simply quote the final results of that analysis:

$$\begin{aligned} S_1 &= \frac{T^2}{8(4\pi)^4} \left\{ \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \left[3 \ln \frac{\Lambda^2}{4\pi T^2} + \frac{17}{6} + \gamma_E + 2 \frac{\zeta'(-1)}{\zeta(-1)} \right] + \right. \\ &+ \left. \frac{9}{2} \left(\ln \frac{\Lambda^2}{4\pi T^2} \right)^2 + \left[\frac{17}{2} + 3\gamma_E + 6 \frac{\zeta'(-1)}{\zeta(-1)} \right] \ln \frac{\Lambda^2}{4\pi T^2} + 48.797635359976(4) \right\} + \mathcal{O}(\epsilon), \end{aligned} \tag{2.23}$$

$$\begin{aligned}
S_2 = & -\frac{T^4}{16(4\pi)^4} \left\{ \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \left[2 \ln \frac{\Lambda^2}{4\pi T^2} + \frac{10}{3} - 2\gamma_E + 4 \frac{\zeta'(-1)}{\zeta(-1)} \right] + \right. \\
& \left. + \left(\ln \frac{\Lambda^2}{4\pi T^2} \right)^2 + \left[\frac{6}{5} - 2\gamma_E + 4 \frac{\zeta'(-3)}{\zeta(-3)} \right] \ln \frac{\Lambda^2}{4\pi T^2} - 25.705543194(2) \right\} - \\
& -\frac{T^4}{512(4\pi)^2} \left\{ \frac{1}{\epsilon} + 4 \ln \frac{\Lambda^2}{4\pi T^2} + 28.92504950930(1) \right\} + \mathcal{O}(\epsilon). \tag{2.24}
\end{aligned}$$

In S_2 , we have on purpose separated one contribution (coming from the part denoted by $S_2^{\text{I,b}}$ in appendix A, cf. eq. (A.57)) from the rest, as it contains an IR singularity, which will not be cancelled by the renormalization of the coupling constant in the full theory, but only by the ultraviolet (UV) singularities originating from the “soft” contributions to the pressure (cf. next section). The numbers in parentheses estimate the numerical uncertainties of the last digits shown.

In order to present the full result, we introduce the following notation for the contributions of various orders to p_E :

$$\Lambda^{2\epsilon} p_E \equiv T^4 \left[\alpha_{E1} + g^2 \alpha_{E2} + \frac{g^4}{(4\pi)^2} \alpha_{E3} + \frac{g^6}{(4\pi)^4} \beta_{E1} + \mathcal{O}(g^8, \epsilon) \right]. \tag{2.25}$$

Here the coefficients α_{Ei}, β_{E1} have been defined in analogy with ref. [4]. They are dimensionless functions of the temperature and of the regularization scale. Inserting eqs. (2.19), (2.20), (2.23) and (2.24) into eq. (2.18), and using everywhere the $\overline{\text{MS}}$ scheme scale parameter $\bar{\Lambda}$, we obtain the following expressions for these coefficients:

$$\alpha_{E1} = \frac{N\pi^2}{90}, \tag{2.26}$$

$$\alpha_{E2} = -\frac{N(N+2)}{3456}, \tag{2.27}$$

$$\alpha_{E3} = \frac{N(N+2)}{10368} \left[(N+8) \ln \frac{\bar{\Lambda}}{4\pi T} + (N+2)\gamma_E + \frac{31}{5} + 12 \frac{\zeta'(-1)}{\zeta(-1)} - 6 \frac{\zeta'(-3)}{\zeta(-3)} \right], \tag{2.28}$$

$$\begin{aligned}
\beta_{E1} = & \frac{N(N+2)(N+8)\pi^2}{41472\epsilon} - \\
& -\frac{N(N+2)}{31104} \left\{ (N+8)^2 \left(\ln \frac{\bar{\Lambda}}{4\pi T} \right)^2 + N^2 \left[2\gamma_E \ln \frac{\bar{\Lambda}}{4\pi T} + \gamma_E^2 + \frac{\zeta(3)}{36} \right] + \right. \\
& + 16N \left[\left(\frac{107}{80} + \frac{5\gamma_E}{4} - \frac{3\pi^2}{8} + \frac{3}{2} \frac{\zeta'(-1)}{\zeta(-1)} - \frac{3}{4} \frac{\zeta'(-3)}{\zeta(-3)} \right) \ln \frac{\bar{\Lambda}}{4\pi T} - 3.9753932807(2) \right] + \\
& \left. + 64 \left[\left(\frac{353}{160} + \frac{\gamma_E}{2} - \frac{3\pi^2}{4} + 3 \frac{\zeta'(-1)}{\zeta(-1)} - \frac{3}{2} \frac{\zeta'(-3)}{\zeta(-3)} \right) \ln \frac{\bar{\Lambda}}{4\pi T} - 10.6970470860(3) \right] \right\}. \tag{2.29}
\end{aligned}$$

The expressions for $\alpha_{E1}, \alpha_{E2}, \alpha_{E3}$ agree with the results of ref. [20] for $N = 1$.

We note from the first line of eq. (2.29) that the fully renormalized result indeed contains an uncancelled $1/\epsilon$ pole. Therefore, something must be wrong with the naive computation that we have carried out. We now turn to the correct procedure for determining $p(T)$ up to $\mathcal{O}(g^6)$.

3. Resummed 4-loop computation

The reason that the computation carried out in the previous section leads to a divergent result, is that it ignores the fact that certain subsets of higher loop graphs amount to generating radiatively a mass for the fields ϕ_i . In the presence of a mass, the results for IR sensitive loop integrals change. This cures the IR problem and leads to the correct weak-coupling expansion. A systematic way to implement such a mass resummation (as well as other resummations, for instance for the quartic coupling) goes via effective field theory methods, as we now review.

At high temperatures, the relevant effective field theory framework is that of dimensional reduction [22, 23]. The basic observation is that the non-zero Matsubara modes are heavy, and certainly cannot cause any IR divergences. Thereby the computation of all static thermodynamic observables can be factorised into two parts: to the contribution from the “hard” momentum modes, $k \sim 2\pi T$, and from the “soft” modes, $k \sim gT$ (the scale $k \sim g^2 T$ does not appear in scalar field theory). For the pressure we will denote the two parts as p_{hard} and p_{soft} , respectively. The effective theory determining p_{soft} reads

$$\mathcal{L}_E = \frac{1}{2} \sum_{j=1}^{3-2\epsilon} \sum_{i=1}^N \partial_j \phi_i \partial_j \phi_i + \frac{1}{2} m_E^2 \sum_{i=1}^N \phi_i \phi_i + \frac{1}{4!} g_E^2 \Lambda^{2\epsilon} \left(\sum_{i=1}^N \phi_i \phi_i \right)^2 + \dots \quad (3.1)$$

A series of infinitely many higher dimensional operators has been truncated given that, as power-counting arguments show (cf. ref. [4]), they cannot contribute to the pressure at $\mathcal{O}(g^6)$. The theory in eq. (3.1) describes the dynamics of the Matsubara zero modes, and thus lives in three dimensions. The parameters here are to be understood as bare parameters. In dimensional regularization, the dimension of ϕ_i is $[\text{GeV}]^{1/2-\epsilon}$ and that of g_E^2 is $[\text{GeV}]^1$. Note that for simplicity we have used the same notation for the fields in eqs. (2.1), (3.1), even though they are independent integration variables, with a different mass dimension. The use of the subscript E is meant to keep in mind the analogy with the effective theory called EQCD in the context of QCD [3]. The determination of the effective parameters (or “matching coefficients”) in eq. (3.1) up to 2-loop level dates back to ref. [24], where the symmetry-breaking phase transition in the case of a massive scalar field was considered; the application of this theory to the computation of the pressure at very high temperatures, where the zero-temperature scalar mass can be ignored, as is done in this paper, was first pursued systematically in ref. [20], as far as we know.

The factorization statement now reads that the physical pressure can be written as [20]

$$p(T) = p_E(T) + p_M(T) = p_{\text{hard}}(T) + p_{\text{soft}}(T) , \quad (3.2)$$

where

$$p_M(T) \equiv \lim_{V \rightarrow \infty} \frac{T}{V} \ln \int \mathcal{D}\phi_i \exp(-S_E) , \quad (3.3)$$

and $S_E = \int d^{3-2\epsilon} \mathbf{x} \mathcal{L}_E$. We adopt a notation in the following whereby the matching coefficient p_E is a bare quantity, like m_E^2 and g_E^2 , while p_{hard} is defined as its $\overline{\text{MS}}$ scheme version. Similarly, p_{soft} is defined to be the $\overline{\text{MS}}$ version of p_M .

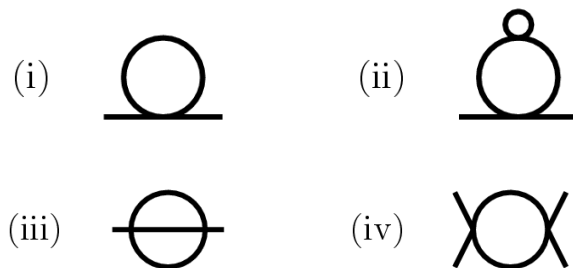


Figure 2: The 1-loop and 2-loop graphs needed for determining the matching coefficients m_E^2 and g_E^2 in eqs. (3.5), (3.6), respectively.

Now, if we were not to carry out any resummation — that is, if the mass parameter m_E^2 in eq. (3.1) were ignored as was the case in the previous section — then the path integral in eq. (3.3) would vanish order by order in dimensional regularization, because the propagators appearing in the computation would contain no mass scales. Therefore, according to eq. (3.2), the computation of the full pressure without any resummation, produces directly the function p_E . In other words, the proper interpretation for the result of the previous section is to treat $p_E(T)$ in eq. (2.25) as an UV matching coefficient: it contains the contributions to the physical pressure from the hard modes, $k \sim 2\pi T$. Interpreted this way, it is IR finite, because the soft contribution $p_M(T)$ has been subtracted (which happens automatically in dimensional regularization by ignoring the mass parameter m_E^2 [20]).

Our task in the remainder of this section is to properly compute p_M , as defined by eq. (3.3). This result can be extracted directly from ref. [14]. Setting $g \rightarrow 0$, $\lambda \rightarrow g_E^2/6$, and $d_A \rightarrow N$ there, we obtain

$$\begin{aligned}
 \frac{\Lambda^{2\epsilon} p_M}{T} = & \frac{m_E^3}{4\pi} \frac{N}{3} \left[1 + \epsilon \left(\ln \frac{\bar{\Lambda}^2}{4m_E^2} + \frac{8}{3} \right) + \mathcal{O}(\epsilon^2) \right] - \\
 & - \frac{g_E^2 m_E^2}{(4\pi)^2} \frac{N(N+2)}{24} \left[1 + 2\epsilon \left(\ln \frac{\bar{\Lambda}^2}{4m_E^2} + 2 \right) + \mathcal{O}(\epsilon^2) \right] - \\
 & - \frac{g_E^4 m_E}{(4\pi)^3} \frac{N(N+2)}{144} \left[\frac{1}{\epsilon} + 3 \ln \frac{\bar{\Lambda}^2}{4m_E^2} + 8 - 4 \ln 2 - \frac{N+2}{2} + \mathcal{O}(\epsilon) \right] + \\
 & + \frac{g_E^6}{(4\pi)^4} \frac{N(N+2)}{1728} \left\{ (N+2) \left[\frac{1}{\epsilon} + 4 \ln \frac{\bar{\Lambda}^2}{4m_E^2} + 4 - 4 \ln 2 \right] - \frac{(N+2)^2}{3} - \right. \\
 & \left. - (N+8) \frac{\pi^2}{24} \left[\frac{1}{\epsilon} + 4 \ln \frac{\bar{\Lambda}^2}{4m_E^2} + 2 + 4 \ln 2 - 84 \frac{\zeta(3)}{\pi^2} \right] + \mathcal{O}(\epsilon) \right\} + \mathcal{O}\left(\frac{g_E^8}{m_E}\right). \quad (3.4)
 \end{aligned}$$

We next have to determine the values of m_E^2 and g_E^2 that appear in eq. (3.4). Following

the notation in ref. [4], these matching coefficients can be written as

$$m_E^2 = T^2 \left\{ g^2 \left[\alpha_{E4} + \alpha_{E5} \epsilon + \mathcal{O}(\epsilon^2) \right] + \frac{g^4}{(4\pi)^2} \left[\alpha_{E6} + \beta_{E2} \epsilon + \mathcal{O}(\epsilon^2) \right] + \mathcal{O}(g^6) \right\}, \quad (3.5)$$

$$g_E^2 = T \left\{ g^2 + \frac{g^4}{(4\pi)^2} \left[\alpha_{E7} + \beta_{E3} \epsilon + \mathcal{O}(\epsilon^2) \right] + \mathcal{O}(g^6) \right\}. \quad (3.6)$$

Up to the orders indicated they are produced by the graphs in figure 2. Using methods explained in some detail in ref. [16], we obtain the values

$$\alpha_{E4} = \frac{N+2}{72}, \quad (3.7)$$

$$\alpha_{E5} = \frac{N+2}{36} \left[\ln \frac{\bar{\Lambda}}{4\pi T} + 1 + \frac{\zeta'(-1)}{\zeta(-1)} \right], \quad (3.8)$$

$$\alpha_{E6} = \frac{N+2}{72} \left[\frac{1}{\epsilon} - \frac{N}{3} \left(\ln \frac{\bar{\Lambda}}{4\pi T} + \gamma_E \right) + \frac{4}{3} \left(\ln \frac{\bar{\Lambda}}{4\pi T} + \frac{3}{2} - \frac{\gamma_E}{2} + \frac{3}{2} \frac{\zeta'(-1)}{\zeta(-1)} \right) \right], \quad (3.9)$$

$$\alpha_{E7} = -\frac{N+8}{3} \left[\ln \frac{\bar{\Lambda}}{4\pi T} + \gamma_E \right], \quad (3.10)$$

$$\begin{aligned} \beta_{E2} = & -\frac{N+2}{1728} \left\{ 24N \ln^2 \frac{\bar{\Lambda}}{4\pi T} + \right. \\ & + 16N \left[\left(1 + 2\gamma_E + \frac{\zeta'(-1)}{\zeta(-1)} \right) \ln \frac{\bar{\Lambda}}{4\pi T} + \frac{\pi^2}{16} + \gamma_E + \gamma_E \frac{\zeta'(-1)}{\zeta(-1)} - \gamma_1 \right] - \\ & - 64 \left[\left(1 - \gamma_E + \frac{\zeta'(-1)}{\zeta(-1)} \right) \ln \frac{\bar{\Lambda}}{4\pi T} + \frac{3}{2} + \frac{\pi^2}{16} - \frac{\gamma_E}{2} - \frac{\gamma_E}{2} \frac{\zeta'(-1)}{\zeta(-1)} + \frac{\gamma_1}{2} + \right. \\ & \left. \left. + \frac{3}{2} \frac{\zeta'(-1)}{\zeta(-1)} + \frac{3}{4} \frac{\zeta''(-1)}{\zeta(-1)} \right] \right\}, \quad (3.11) \end{aligned}$$

$$\beta_{E3} = -\frac{N+8}{3} \left[\ln^2 \frac{\bar{\Lambda}}{4\pi T} + 2\gamma_E \ln \frac{\bar{\Lambda}}{4\pi T} + \frac{\pi^2}{8} - 2\gamma_1 \right]. \quad (3.12)$$

Here γ_1 is a Stieltjes constant, defined through the series $\zeta(s) = 1/(s-1) + \sum_{n=0}^{\infty} \gamma_n (-1)^n (s-1)^n / n!$. The values of α_{E4}, α_{E6} agree with the results of ref. [20] for $N=1$. The value of α_{E7} agrees with the results of ref. [24], for $N=1, 2, 4$.¹ In the following, we denote by $\alpha_{Ei}^{\overline{\text{MS}}}, \beta_{Ei}^{\overline{\text{MS}}}$ couplings from which the $1/\epsilon$ -divergences have been subtracted; this is in fact relevant only for α_{E6} and β_{E1} , cf. eqs. (2.29), (3.9).

We note from eqs. (3.5), (3.9) that the mass parameter m_E^2 has a divergent part. From the point of view of the effective theory, this divergence acts as a counterterm,

$$\delta m_E^2 = \frac{g^4 T^2}{(4\pi)^2} \frac{N+2}{72\epsilon}. \quad (3.13)$$

Indeed, this agrees with the counterterm that can be determined within the effective theory, by just requiring renormalizability of \mathcal{L}_E [24]. It will be convenient for the following to also define finite parameters from which the divergence as well as terms proportional to ϵ have

¹There is a sign error in eq. (60) of ref. [24], relevant for the case $N=1$.

been subtracted; we denote these by

$$\hat{m}_3^2(\bar{\Lambda}) \equiv \frac{m_3^2(\bar{\Lambda})}{T^2} \equiv g^2 \alpha_{E4} + \frac{g^4}{(4\pi)^2} \alpha_{E6}^{\overline{\text{MS}}}, \quad (3.14)$$

$$\hat{g}_3^2 \equiv g^2 + \frac{g^4}{(4\pi)^2} \alpha_{E7}. \quad (3.15)$$

The parameter \hat{g}_3^2 is renormalization group (RG) invariant up to the order computed, while $\hat{m}_3^2(\bar{\Lambda})$ has dependence at order g^4 , because the counterterm in eq. (3.13) has been subtracted.

Re-expanding now the bare expression of eq. (3.4) by treating the counterterm in eq. (3.13) as a perturbation, we obtain a “renormalized” expression for p_M :

$$\frac{\Lambda^{2\epsilon} p_M}{T^4} = -\frac{g^6}{(4\pi)^4} \frac{N(N+2)(N+8)\pi^2}{41472\epsilon} + \frac{\Lambda^{2\epsilon} p_{\text{soft}}}{T^4}, \quad (3.16)$$

where, after sending $\epsilon \rightarrow 0$,

$$\begin{aligned} \frac{p_{\text{soft}}}{T^4} &= \frac{\hat{m}_3^3(\bar{\Lambda})}{4\pi} \frac{N}{3} - \frac{\hat{g}_3^2 \hat{m}_3^2(\bar{\Lambda})}{(4\pi)^2} \frac{N(N+2)}{24} - \\ &\quad - \frac{\hat{g}_3^4 \hat{m}_3(\bar{\Lambda})}{(4\pi)^3} \frac{N(N+2)}{288} \left[8 \ln \frac{\bar{\Lambda}}{m_3(\bar{\Lambda})} + 10 - 16 \ln 2 - N \right] + \\ &\quad + \frac{\hat{g}_3^6}{(4\pi)^4} \frac{N(N+2)}{864} \left\{ (N+2) \left[2 \ln \frac{\bar{\Lambda}}{m_3(\bar{\Lambda})} + 1 - 4 \ln 2 \right] - \frac{(N+2)^2}{6} - \right. \\ &\quad \left. - (N+8) \frac{\pi^2}{24} \left[4 \ln \frac{\bar{\Lambda}}{m_3(\bar{\Lambda})} + 1 - 2 \ln 2 - 42 \frac{\zeta(3)}{\pi^2} \right] \right\} + \mathcal{O}\left(\frac{\hat{g}_3^8}{\hat{m}_3(\bar{\Lambda})}\right). \end{aligned} \quad (3.17)$$

We note from eq. (3.16) that the only divergence appearing in p_M after the re-expansion of m_E^2 is of $\mathcal{O}(g^6)$. Therefore, none of the coefficients $\alpha_{E5}, \beta_{E2}, \beta_{E3}$ in eqs. (3.5), (3.6), which multiply terms of order $\mathcal{O}(\epsilon)$, play a role in p_{soft} at $\mathcal{O}(g^6)$. In this respect, the present theory differs from QCD, where the corresponding coefficients do play a role; the reason for the difference is that in QCD there is an $1/\epsilon$ -divergence in the second term of eq. (3.4).

To conclude this section, we remark that it may be convenient, following ref. [12], to also express p_E in terms of the parameter \hat{g}_3^2 . This implements a certain (arbitrary) resummation; the practical effect is minor (in fact, for the scale choice that we will make in eq. (4.16), which leads to the vanishing of α_{E7} , there is no effect at all, cf. eq. (3.19)), and we present this last step only in order to allow for a more compact numerical handling of the result. From eqs. (2.25), (2.29), we obtain

$$\frac{\Lambda^{2\epsilon} p_E}{T^4} = \frac{g^6}{(4\pi)^4} \frac{N(N+2)(N+8)\pi^2}{41472\epsilon} + \frac{\Lambda^{2\epsilon} p_{\text{hard}}}{T^4}, \quad (3.18)$$

where, after setting $\epsilon \rightarrow 0$, the finite function p_{hard} can be written as

$$\begin{aligned} \frac{p_{\text{hard}}}{T^4} &= \alpha_{E1} + \hat{g}_3^2 \alpha_{E2} + \frac{\hat{g}_3^4}{(4\pi)^2} (\alpha_{E3} - \alpha_{E2} \alpha_{E7}) + \\ &\quad + \frac{\hat{g}_3^6}{(4\pi)^4} \left[\beta_{E1}^{\overline{\text{MS}}} + 2\alpha_{E2} \alpha_{E7}^2 - 2\alpha_{E3} \alpha_{E7} \right] + \mathcal{O}(\hat{g}_3^8). \end{aligned} \quad (3.19)$$

In the numerical results of the next section, we refer to the various orders of the weak-coupling expansion according to the power of \hat{m}_3, \hat{g}_3 that appear, with the rule $\mathcal{O}(\hat{m}_3) = \mathcal{O}(\hat{g}_3) = \mathcal{O}(g)$. In other words, “ $\mathcal{O}(g^n)$ ” denotes $\mathcal{O}(\hat{g}_3^{n-k} \hat{m}_3^k)$ in the expression constituted by the sum of (3.17), (3.19). If \hat{m}_3, \hat{g}_3 were to be re-expanded in terms of g , one would recover the strict weak-coupling expansion (given in eq. (4.1) below); however, it is useful to keep the result in an unexpanded form, because this makes it more manageable, and because the unexpanded form introduces resummations of higher order contributions which may be numerically significant for the slowly convergent part p_{soft} in eq. (3.17) [24, 20, 3, 4, 25] (in practice, though, the effects caused by this resummation are not dramatic).

4. Results and discussion

Inserting eqs. (3.5), (3.6) into eq. (3.4), summing together with eq. (2.25), and sending $\epsilon \rightarrow 0$, we obtain the strict weak-coupling expansion for the pressure of our theory:

$$\begin{aligned}
 \frac{p(T)}{T^4} = & \alpha_{\text{E1}} + g^2 \alpha_{\text{E2}} + \frac{g^3 N}{4\pi} \alpha_{\text{E4}}^{3/2} + \frac{g^4}{(4\pi)^2} \left[\alpha_{\text{E3}} - \frac{N(N+2)}{24} \alpha_{\text{E4}} \right] + \\
 & + \frac{g^5}{(4\pi)^3} \frac{N}{2} \alpha_{\text{E4}}^{1/2} \left[\alpha_{\text{E6}}^{\overline{\text{MS}}} - \frac{N+2}{144} \left(8 \ln \frac{\bar{\Lambda}}{gT \sqrt{\alpha_{\text{E4}}}} + 10 - 16 \ln 2 - N \right) \right] + \\
 & + \frac{g^6}{(4\pi)^4} \left\{ \beta_{\text{E1}}^{\overline{\text{MS}}} - \frac{N(N+2)}{24} \left[\alpha_{\text{E6}}^{\overline{\text{MS}}} + \alpha_{\text{E4}} \alpha_{\text{E7}} - \right. \right. \\
 & \left. \left. - \frac{N+2}{36} \left(2 \ln \frac{\bar{\Lambda}}{gT \sqrt{\alpha_{\text{E4}}}} + 1 - 4 \ln 2 \right) + \frac{(N+2)^2}{216} + \right. \right. \\
 & \left. \left. + \frac{(N+8)\pi^2}{864} \left(4 \ln \frac{\bar{\Lambda}}{gT \sqrt{\alpha_{\text{E4}}}} + 1 - 2 \ln 2 - 42 \frac{\zeta(3)}{\pi^2} \right) \right] \right\} + \mathcal{O}(g^7). \quad (4.1)
 \end{aligned}$$

We note that the $1/\epsilon$ -divergences in eqs. (3.16), (3.18) have cancelled against each other, as must be the case for a consistently computed physical quantity. Inserting the expansions from eqs. (2.26)–(2.29) and eqs. (3.7)–(3.10), finally yields the explicit expression

$$\frac{p(T)}{T^4} = \frac{\pi^2 N}{90} \sum_{i=0}^6 p_i \left(\frac{g}{4\pi} \right)^i, \quad (4.2)$$

where $g \equiv [g^2(\bar{\Lambda})]^{1/2}$, and the coefficients read

$$p_0 = 1, \quad (4.3)$$

$$p_1 = 0, \quad (4.4)$$

$$p_2 = -\frac{5}{12}(N+2), \quad (4.5)$$

$$p_3 = \frac{5\sqrt{2}}{9}(N+2)^{\frac{3}{2}}, \quad (4.6)$$

$$p_4 = \frac{5}{36}(N+2) \left\{ N \left[\ln \frac{\bar{\Lambda}}{4\pi T} + \gamma_{\text{E}} - 6 \right] + 8 \left[\ln \frac{\bar{\Lambda}}{4\pi T} - \frac{29}{40} + \frac{\gamma_{\text{E}}}{4} + \frac{3}{2} \frac{\zeta'(-1)}{\zeta(-1)} - \frac{3}{4} \frac{\zeta'(-3)}{\zeta(-3)} \right] \right\}, \quad (4.7)$$

$$\begin{aligned}
 p_5 = & -\frac{5}{9\sqrt{2}}(N+2)^{\frac{3}{2}} \left\{ -12 \ln \left(\frac{g}{\pi} \sqrt{\frac{N+2}{72}} \right) + \right. \\
 & \left. + N \left[\ln \frac{\bar{\Lambda}}{4\pi T} + \gamma_E - \frac{3}{2} \right] + 8 \left[\ln \frac{\bar{\Lambda}}{4\pi T} + \frac{9}{8} + \frac{\gamma_E}{4} - \frac{3}{4} \frac{\zeta'(-1)}{\zeta(-1)} \right] \right\}, \quad (4.8)
 \end{aligned}$$

$$\begin{aligned}
 p_6 = & -\frac{5}{108}(N+2) \left\{ \left[72(N+2) - 6(N+8)\pi^2 \right] \ln \left(\frac{g}{\pi} \sqrt{\frac{N+2}{72}} \right) + (N+8)^2 \left(\ln \frac{\bar{\Lambda}}{4\pi T} \right)^2 + \right. \\
 & + N^2 \left[\left(2\gamma_E - 12 \right) \ln \frac{\bar{\Lambda}}{4\pi T} + 6 - 12\gamma_E + \gamma_E^2 + \frac{\zeta(3)}{36} \right] + \\
 & + 16N \left[\left(-\frac{493}{80} + \frac{5\gamma_E}{4} + \frac{3}{2} \frac{\zeta'(-1)}{\zeta(-1)} - \frac{3}{4} \frac{\zeta'(-3)}{\zeta(-3)} \right) \ln \frac{\bar{\Lambda}}{4\pi T} - 0.9991160242(2) \right] + \\
 & \left. + 64 \left[\left(-\frac{127}{160} + \frac{\gamma_E}{2} + 3 \frac{\zeta'(-1)}{\zeta(-1)} - \frac{3}{2} \frac{\zeta'(-3)}{\zeta(-3)} \right) \ln \frac{\bar{\Lambda}}{4\pi T} - 9.0905637831(3) \right] \right\}. \quad (4.9)
 \end{aligned}$$

A number of simple crosschecks can be made on eqs. (4.2)–(4.9). Making use of the RG equation,

$$\bar{\Lambda} \frac{dg}{d\bar{\Lambda}} = \frac{g^3}{(4\pi)^2} \frac{N+8}{6} - \frac{g^5}{(4\pi)^4} \frac{3N+14}{6} + \mathcal{O}(g^7), \quad (4.10)$$

it is easy to verify that the result is RG invariant up to the order computed. Setting $N = 1$, terms up to $\mathcal{O}(g^5)$ agree with refs. [1, 19, 20]. Finally, taking the limit $N \rightarrow \infty$ with $g^2 N$ fixed, we get

$$\begin{aligned}
 \frac{p(T)}{T^4} \approx & \frac{\pi^2 N}{90} \left\{ 1 + \mathcal{G}^2 \left(-\frac{5}{12} \right) + \mathcal{G}^3 \left(\frac{5\sqrt{2}}{9} \right) + \right. \\
 & + \mathcal{G}^4 \left(\frac{5}{36} \right) \left(\ln \frac{\bar{\Lambda}}{4\pi T} + \gamma_E - 6 \right) + \mathcal{G}^5 \left(-\frac{5}{9\sqrt{2}} \right) \left(\ln \frac{\bar{\Lambda}}{4\pi T} + \gamma_E - \frac{3}{2} \right) + \\
 & \left. + \mathcal{G}^6 \left(-\frac{5}{108} \right) \left[\left(\ln \frac{\bar{\Lambda}}{4\pi T} \right)^2 + 2(\gamma_E - 6) \ln \frac{\bar{\Lambda}}{4\pi T} + 6 - 12\gamma_E + \gamma_E^2 + \frac{\zeta(3)}{36} \right] \right\}, \quad (4.11)
 \end{aligned}$$

where $\mathcal{G} \equiv g\sqrt{N}/4\pi$. This agrees with eq. (5.8) of ref. [27]; unfortunately most of the non-trivial structures, like logarithms of g or the genuine 4-loop sum-integrals that we were only able to determine numerically, disappear in the large- N limit.

Let us finally evaluate our result numerically. Though the effect is moderate in practice, we reiterate that we find it convenient not to use eq. (4.1) for the numerical evaluation, but the unexpanded expression $p_{\text{hard}} + p_{\text{soft}}$, defined as a sum of eqs. (3.17), (3.19).

If we want to present the numerical results as a function of T , we first have to insert a value for the renormalized quartic coupling g^2 as a function of the scale $\bar{\Lambda}$, because the temperature-dependence emerges in connection with logarithms related to the running of g^2 (cf. eqs. (4.2)–(4.9)). Defining $b_1 \equiv |\beta_1|/(4\pi)^2$ and $b_2 \equiv |\beta_2|/(4\pi)^4$, where β_1, β_2 are from eq. (2.3), as well as $\alpha \equiv b_2/b_1$ and $t \equiv 2b_1 \ln(\bar{\Lambda}_{\text{Landau}}/\bar{\Lambda})$, the 2-loop RG-equation reads

$$\frac{dg^2}{dt} = -g^4 + \alpha g^6. \quad (4.12)$$

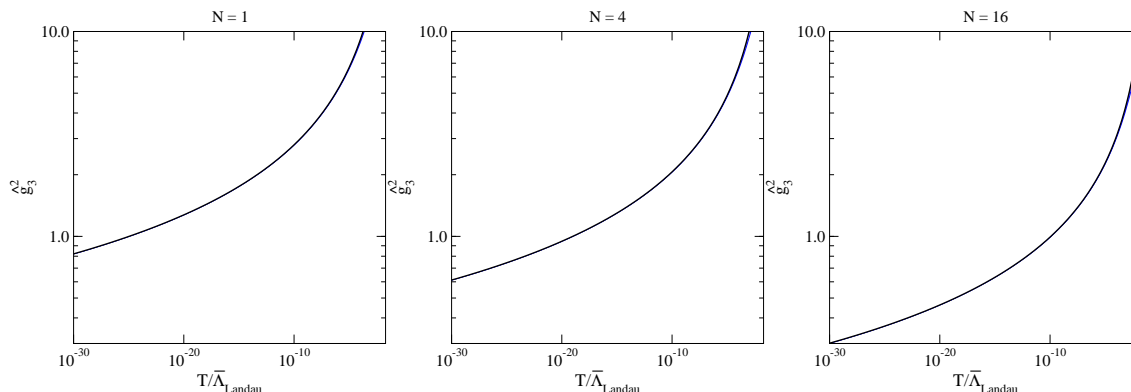


Figure 3: The effective gauge coupling in eq. (3.15), as function of $T/\bar{\Lambda}_{\text{Landau}}$, where $\bar{\Lambda}_{\text{Landau}}$ is defined in eq. (4.13). From left to right, $N = 1, 4, 16$. The scale $\bar{\Lambda}$ has been varied within the range $(0.5 \dots 2.0)\bar{\Lambda}_{\text{opt}}$ (the grey band), with $\bar{\Lambda}_{\text{opt}}$ defined in eq. (4.16), but the effects are almost invisible on our resolution.

This equation can be solved up to a boundary condition. In formal analogy with QCD, we define the boundary condition such that

$$\bar{\Lambda}_{\text{Landau}} \equiv \lim_{\bar{\Lambda} \rightarrow 0} \bar{\Lambda} \left[b_1 g^2 \right]^{-b_2/2b_1} \exp \left[\frac{1}{2b_1 g^2} \right]. \quad (4.13)$$

The solution then reduces to the equation

$$\frac{1}{g^2} + \alpha \ln \left(\frac{1}{g^2} - \alpha \right) = t + \alpha \ln b_1, \quad (4.14)$$

which for small $\bar{\Lambda}$ yields the approximate behaviour

$$\frac{1}{g^2} \approx 2b_1 \ln \frac{\bar{\Lambda}_{\text{Landau}}}{\bar{\Lambda}} - \frac{b_2}{b_1} \ln \left(2 \ln \frac{\bar{\Lambda}_{\text{Landau}}}{\bar{\Lambda}} \right). \quad (4.15)$$

Moreover, following ref. [26], we define an “optimal” scale according to the simultaneous 1-loop “fastest apparent convergence” and “principal of minimal sensitivity” point obtained for the effective coupling \hat{g}_3^2 , defined in eq. (3.15):

$$\bar{\Lambda}_{\text{opt}} \equiv 4\pi e^{-\gamma_E} T. \quad (4.16)$$

The scale $\bar{\Lambda}$ will then be varied in the range $(0.5 \dots 2.0)\bar{\Lambda}_{\text{opt}}$ around this point. Note that the scale choice in eq. (4.16) leads also to a formal simplification of the pressure; for instance the expression in eq. (4.11) obtains the form

$$\frac{p(T)}{T^4} \approx \frac{\pi^2 N}{90} \left\{ 1 - \frac{5}{12} \left[\mathcal{G}^2 - \frac{4\sqrt{2}}{3} \mathcal{G}^3 + 2\mathcal{G}^4 - \sqrt{2} \mathcal{G}^5 + \left(\frac{2}{3} + \frac{\zeta(3)}{324} \right) \mathcal{G}^6 \right] \right\}. \quad (4.17)$$

We now first plot \hat{g}_3^2 as a function of $T/\bar{\Lambda}_{\text{Landau}}$ and N . The results are shown in figure 3. The scale $\bar{\Lambda}$ is chosen according to eq. (4.16) but the dependence on the choice is so small within the range mentioned that it is almost invisible.

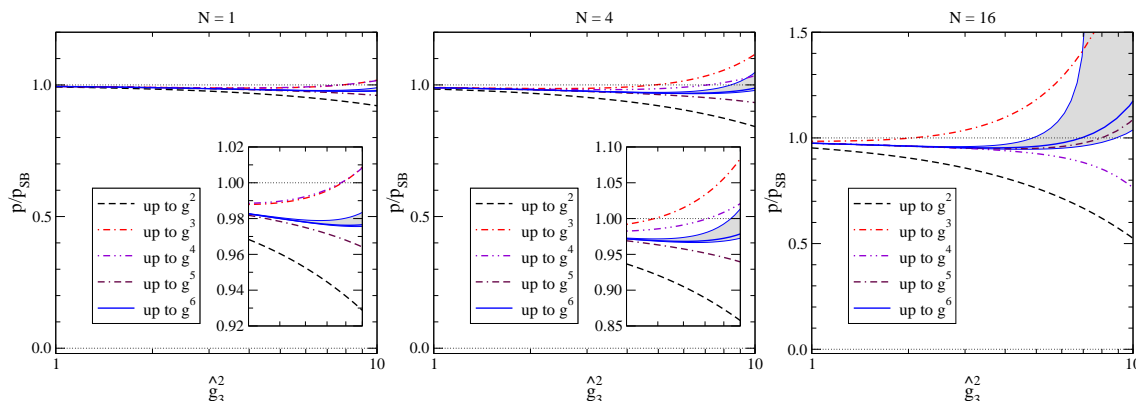


Figure 4: The resummed perturbative result, $p(T) = p_{\text{hard}}(T) + p_{\text{soft}}(T)$, normalised to the free Stefan-Boltzmann result denoted by p_{SB} , as a function of the effective gauge coupling shown in figure 3. From left to right, $N = 1, 4, 16$. For the contribution of $\mathcal{O}(g^6)$, the scale $\bar{\Lambda}$ has been varied within the range $(0.5 \dots 2.0)\bar{\Lambda}_{\text{opt}}$ (the grey band), with $\bar{\Lambda}_{\text{opt}}$ defined in eq. (4.16).

Finally we plot the pressure, normalised to the free result, in figure 4. We have used an overall resolution as would be relevant for QCD, where one hopes to reach an accuracy on the 10% level or so; the inserts display finer structures that are invisible on this resolution but may still be of some academic interest. Since $T/\bar{\Lambda}_{\text{Landau}}$ is a quantity with which it is difficult to associate anything in QCD, we choose to use \hat{g}_3^2 as the horizontal axis in this figure. If desired, the conversion to $T/\bar{\Lambda}_{\text{Landau}}$ goes through figure 3.

It can be seen that the patterns in figure 4 are somewhat similar to those familiar from QCD: the order $\mathcal{O}(g^3)$ strongly overshoots the free result, and the subsequent orders then slowly converge towards a common value. However, for $N = 16$, i.e. the case with the same number of scalar degrees of freedom as QCD, control is certainly lost once $\hat{g}_3^2 \gg 4.0$. It is perhaps also worth stressing that even at $N = 16$, the results deviate fairly substantially from eq. (4.17); relative corrections to the large- N limit can be as large as $\sim 8/N$, cf. eq. (4.9).

A perhaps more satisfying view on the result can be obtained if the hard and the soft contributions to the pressure, p_{hard} and p_{soft} , are plotted separately. This has been done in figure 5 for $N = 16$, where the absolute values of each new order are shown. It is clear that at least the expansion for p_{hard} does appear to converge up to $\hat{g}_3^2 \sim 5.0$; for p_{soft} the situation is worse, since the $\mathcal{O}(g^6)$ result is larger than the $\mathcal{O}(g^5)$ result even at fairly small \hat{g}_3^2 . This is, however, due to the fact that the $\mathcal{O}(g^5)$ result crosses zero at $\hat{g}_3^2 \approx 2.5$, and does not necessarily signal a total breakdown of the series.

5. Conclusions

We have computed in this paper the pressure of $O(N)$ scalar field theory up to order g^6 in the weak-coupling expansion. In terms of the loop expansion, this corresponds to the

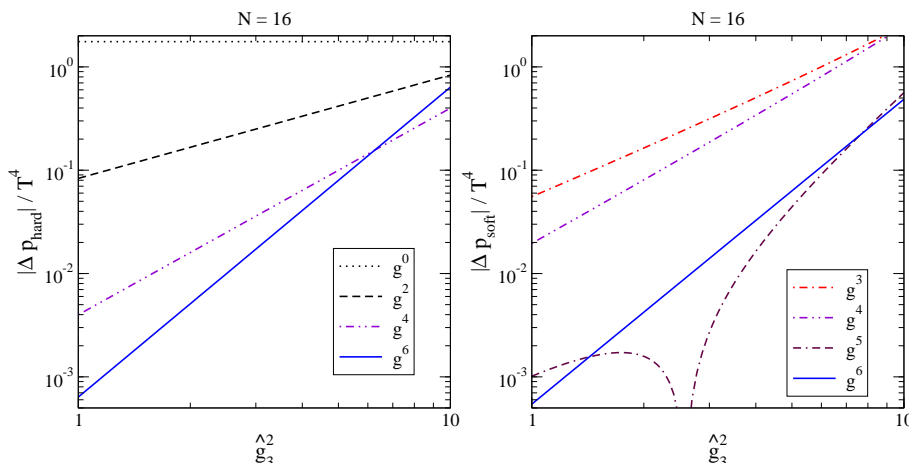


Figure 5: The absolute values of the various order contributions to p_{hard}/T^4 (left) and p_{soft}/T^4 (right), as a function of the effective gauge coupling shown in figure 3, for $N = 16$. The convergence appears to be better for p_{hard}/T^4 , but p_{soft}/T^4 shows some convergence as well.

inclusion of all 4-loop diagrams, as well as infinite subsets of higher-loop diagrams needed in order to cancel the infrared divergence of the naive perturbative computation.

The main motivation for this paper has been “technological”: we have demonstrated that 4-loop sum-integrals are doable, with divergent parts and logarithms handled analytically, and constant parts evaluated numerically. The essential ingredients allowing for the evaluation of the single genuine 4-loop sum-integral that appeared in our computation, denoted by S_2 , were the realization that renormalizing the theory before carrying out the sum-integral, allows to simplify the structure that needs to be considered (cf. eq. (2.22)), as well as an application of mixed coordinate and momentum-space techniques (cf. appendix A.3).

With further work, the constant parts of the 4-loop pressure (the numerical values in eq. (4.9)) might also be doable analytically. This, however, is not necessary from the QCD point of view, where the 4-loop contribution in any case involves a non-perturbative term [5, 6] that can only be determined numerically [7, 8].

The complete result, up to $\mathcal{O}(g^6)$, is shown in eqs. (4.2)–(4.9). A more compact form, resumming hard contributions, is given by a combination of eqs. (3.17), (3.19), with coefficients given in eqs. (3.7)–(3.10) and (2.26)–(2.29).

Though we have thus demonstrated the feasibility of going to 4-loop level in thermal field theory, we would at the same time like to stress that the present computation was based on a “brute force” approach for the evaluation of the 3-loop and 4-loop sum-integrals (cf. appendix A). Such an approach requires a lot of patience, and is simultaneously susceptible to errors. Thinking about the case of QCD, where a much larger set of genuine 4-loop integrals will appear, it would clearly be most desirable to develop somewhat more automated techniques for the evaluation of the loop integrals, in analogy with what has been achieved in recent years for massive vacuum integrals at zero temperature [28, 29], in

order to carry out the computation in a more controllable fashion.

Finally, we should point out that the model we have considered has also been a popular testing ground for many different theoretical tools as well as improved approximation schemes [30]–[47], in the latter case with the goal of learning something about their numerical convergence. We have shown here that, for $N = 16$, the resummed weak-coupling expansion does appear to show some convergence up to the point where the scale-invariant effective coupling constant \hat{g}_3^2 , defined in eq. (3.15), reaches values $\hat{g}_3^2 \sim 4.0$. This includes all values relevant for QCD above the temperature $T \sim \Lambda_{\overline{\text{MS}}}$ [16], and therefore should perhaps be interpreted as a positive feature from the point of view of the applicability of resummed perturbation theory at high temperatures, even though it of course must be stressed that such a direct comparison between scalar field theory and QCD is far too naive.

Acknowledgments

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A. Details of the computation

A.1 Self-energies

A.1.1 $\Pi(P)$

Before attacking the two sum-integrals in eqs. (2.21), (2.22), let us review a few straightforwardly verifiable properties of the functions $\Pi(P)$, and $\bar{\Pi}(P)$, defined in eqs. (2.8), (2.9). [We will also define a third similar function in section A.1.2, denoted by $\tilde{\Pi}(P)$.] As has been shown in ref. [1], the first of these can be written in the form

$$\Pi(P) = \Pi^{(0)}(P) + \frac{2\mathcal{I}_1}{P^2} + \Delta\Pi(P), \tag{A.1}$$

where

$$\Pi^{(0)}(P) \equiv \frac{\beta}{(P^2)^\epsilon} = \frac{\Lambda^{2\epsilon}}{(4\pi)^{2-\epsilon}} \frac{\Gamma(\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)} \frac{1}{(P^2)^\epsilon} \tag{A.2}$$

denotes its zero-temperature limit and is responsible for its leading UV ($P^2 \rightarrow \infty$) behaviour. The leading UV behaviour of the rest, i.e. the finite-temperature part of the function, is on the other hand obtainable by letting P become arbitrarily large in either of the two propagators of $\Pi(P)$ and then integrating over the other, which produces $2\mathcal{I}_1/P^2$. Subsequently, the remaining part of the function, denoted here by $\Delta\Pi(P)$, behaves in the UV as $1/P^4$.

At $\epsilon = 0$, the UV-finite subtracted function $\Delta\Pi(P)$ can be written in a simple form by using a three-dimensional (spatial) Fourier transform. This gives [1]

$$\Delta\Pi(P) = \frac{T}{(4\pi)^2} \int d^3\mathbf{r} \frac{1}{r^2} e^{i\mathbf{p}\cdot\mathbf{r}} e^{-|p_0|r} \left(\coth \bar{r} - \frac{1}{\bar{r}} - \frac{\bar{r}}{3} \right), \quad (\text{A.3})$$

where $\bar{r} \equiv 2\pi T r$.

Sometimes we need to refer to the finite-temperature part of $\Pi(P)$, defined as $\Pi^{(T)}(P) \equiv \Pi(P) - \Pi^{(0)}(P)$. Its Fourier representation after setting $\epsilon \rightarrow 0$ is obtained from eq. (A.3) by leaving out the last term $-\bar{r}/3$ from inside the parentheses. In one instance, we will also require another version of $\Delta\Pi(P)$, in which its leading UV behaviour is further subtracted. This function is defined as

$$\Delta'\Pi(P) \equiv \Delta\Pi(P) - \frac{8T^4 J_1}{3-2\epsilon} \left(\frac{1}{P^4} - (4-2\epsilon) \frac{p_0^2}{P^6} \right) (1 - \delta_{p_0}), \quad (\text{A.4})$$

with the Kronecker- δ ($\delta_{p_0} \equiv \delta_{p_0,0}$) introduced in the last term for convenience. Its Fourier representation after $\epsilon \rightarrow 0$ is equivalent to eq. (A.3) apart from having an extra term of the form $+(1 - \delta_{p_0}) \bar{r}^3/45$ inside the parentheses. The constant J_1 is evaluated in ref. [1] and reads

$$J_1 = 2^{-2+2\epsilon} \pi^{-3/2+\epsilon} \left(\frac{\Lambda^2}{T^2} \right)^\epsilon \frac{\Gamma(4-2\epsilon)}{\Gamma(3/2-\epsilon)} \zeta(4-2\epsilon). \quad (\text{A.5})$$

A.1.2 $\tilde{\Pi}(P)$

It will be convenient in the following to add a third function to those given in eqs. (2.8), (2.9). We define this function through

$$\tilde{\Pi}(P) \equiv \not\int_Q \frac{\Pi^{(0)}(Q)}{(Q-P)^2}, \quad (\text{A.6})$$

where $\Pi^{(0)}(Q)$ denotes the zero-temperature part of $\Pi(Q)$ (cf. eq. (A.2)).

An analogous reasoning as for $\Pi(P)$ produces for $\tilde{\Pi}(P)$ the representation

$$\tilde{\Pi}(P) = \tilde{\Pi}^{(0)}(P) + \mathcal{I}_1 \Pi^{(0)}(P) + \Delta\tilde{\Pi}(P), \quad (\text{A.7})$$

where

$$\tilde{\Pi}^{(0)}(P) \equiv \tilde{\beta}(P^2)^{1-2\epsilon} = \frac{\Lambda^{2\epsilon}}{(4\pi)^{2-\epsilon}} \frac{\Gamma(1-\epsilon)\Gamma(2-2\epsilon)\Gamma(-1+2\epsilon)}{\Gamma(\epsilon)\Gamma(3-3\epsilon)} \beta(P^2)^{1-2\epsilon}, \quad (\text{A.8})$$

and $\Delta\tilde{\Pi}(P)$ behaves in the UV like $1/P^2$. The only difference with respect to the previous calculation is that when taking the UV limit of the finite-temperature part of $\tilde{\Pi}(P)$, the two ‘‘propagators’’ of which $\tilde{\Pi}(P)$ is composed are not symmetric, and the dominant $P^2 \rightarrow \infty$ behaviour is obtained when the large momentum is routed solely through the function $\Pi^{(0)}(Q)$, while the argument of the $1/(Q-P)^2$ propagator is integrated over.

As with $\Pi(P)$, we next derive a spatial Fourier representation for $\Delta\tilde{\Pi}(P)$. We will carry out the Fourier-transforms strictly in three dimensions, even though the functions

transformed may contain $\epsilon \neq 0$; this is sufficient since, as we will see, no divergences in ϵ appear in the formal coordinate-space representations (there are in fact divergences hidden in the lower ranges of the \mathbf{r} -integration in individual terms, but they cancel for the finite quantity $\Delta\tilde{\Pi}(P)$ that we are ultimately interested in). Defining the (inverse) Fourier transform

$$\mathcal{F}(q_0, r; \alpha) \equiv \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{e^{-i\mathbf{q}\cdot\mathbf{r}}}{(q^2 + q_0^2)^\alpha} = \frac{2^{-1/2-\alpha}\pi^{-3/2}}{\Gamma(\alpha)} \left(\frac{|q_0|}{r}\right)^{3/2-\alpha} K_{3/2-\alpha}(|q_0|r), \quad (\text{A.9})$$

and making use of eqs. (A.2), (A.8), we obtain the Fourier-representations

$$\begin{aligned} \frac{1}{(Q-P)^2} &= \int d^3\mathbf{r} e^{i(\mathbf{q}-\mathbf{p})\cdot\mathbf{r}} \mathcal{F}(q_0 - p_0, r; 1) \\ &= \int d^3\mathbf{r} e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{r}} e^{-|q_0-p_0|r} \frac{1}{4\pi r}, \end{aligned} \quad (\text{A.10})$$

$$\begin{aligned} \Pi^{(0)}(Q) &\approx \int d^3\mathbf{r} e^{i\mathbf{q}\cdot\mathbf{r}} \frac{\Lambda^{2\epsilon}}{(4\pi)^{2-\epsilon}} \frac{\Gamma^2(1-\epsilon)\Gamma(\epsilon)}{\Gamma(2-2\epsilon)} \mathcal{F}(q_0, r; \epsilon) \\ &\approx \int d^3\mathbf{r} e^{i\mathbf{q}\cdot\mathbf{r}} \left\{ e^{-|q_0|r} \frac{2}{(4\pi r)^3} (1 + |q_0|r) + \mathcal{O}(\epsilon) \right\}, \end{aligned} \quad (\text{A.11})$$

$$\begin{aligned} \tilde{\Pi}^{(0)}(Q) &\approx \int d^3\mathbf{r} e^{i\mathbf{q}\cdot\mathbf{r}} \frac{\Lambda^{4\epsilon}}{(4\pi)^{4-2\epsilon}} \frac{\Gamma^3(1-\epsilon)\Gamma(-1+2\epsilon)}{\Gamma(3-3\epsilon)} \mathcal{F}(q_0, r; -1+2\epsilon) \\ &\approx \int d^3\mathbf{r} e^{i\mathbf{q}\cdot\mathbf{r}} \left\{ e^{-|q_0|r} \frac{2}{(4\pi r)^5} (3 + 3|q_0|r + q_0^2 r^2) + \mathcal{O}(\epsilon) \right\}, \end{aligned} \quad (\text{A.12})$$

where the symbol “ \approx ” is a reminder of the fact that the integrals are not well-defined around the origin, and we note that the divergent factors $\Gamma(\epsilon)$ and $\Gamma(-1+2\epsilon)$ have cancelled against the corresponding ones in eq. (A.9).

Inserting now eqs. (A.10), (A.11) into the definition of $\tilde{\Pi}(P)$ in eq. (A.6), produces

$$\begin{aligned} \tilde{\Pi}(P) &\approx \frac{2T}{(4\pi)^4} \int d^3\mathbf{r} e^{i\mathbf{p}\cdot\mathbf{r}} \left\{ \frac{1}{r^4} \sum_{q_0} (1 + |q_0|r) e^{-|q_0|r} e^{-|q_0-p_0|r} + \mathcal{O}(\epsilon) \right\} \\ &\approx \frac{T}{(4\pi)^4} \int d^3\mathbf{r} e^{i\mathbf{p}\cdot\mathbf{r}} \left\{ e^{-|p_0|r} \frac{1}{r^4} \left[\bar{r} \text{csch}^2 \bar{r} + (2 + |\bar{p}_0|\bar{r})(|\bar{p}_0| + \coth \bar{r}) \right] + \mathcal{O}(\epsilon) \right\}, \end{aligned} \quad (\text{A.13})$$

with $\bar{p}_0 \equiv p_0/2\pi T$. Subtracting eq. (A.12) as well as a result obtained from eq. (A.11),

$$\mathcal{I}_1 \Pi^{(0)}(P) \approx \frac{T}{(4\pi)^4} \int d^3\mathbf{r} e^{i\mathbf{p}\cdot\mathbf{r}} \left\{ e^{-|p_0|r} \frac{1}{r^4} \frac{\bar{r}}{3} (1 + |\bar{p}_0|\bar{r}) + \mathcal{O}(\epsilon) \right\}, \quad (\text{A.14})$$

leads to the well-defined form

$$\begin{aligned} \Delta\tilde{\Pi}(P)|_{\epsilon=0} &= \frac{T}{(4\pi)^4} \int d^3\mathbf{r} \frac{1}{r^4} e^{i\mathbf{p}\cdot\mathbf{r}} e^{-|p_0|r} \left\{ \bar{r} \text{csch}^2 \bar{r} + (2 + |\bar{p}_0|\bar{r})(|\bar{p}_0| + \coth \bar{r}) - \right. \\ &\quad \left. - \frac{1}{\bar{r}} (3 + 3|\bar{p}_0|\bar{r} + \bar{p}_0^2 \bar{r}^2) - \frac{\bar{r}}{3} (1 + |\bar{p}_0|\bar{r}) \right\}, \end{aligned} \quad (\text{A.15})$$

where the integration is convergent around the origin such that we have replaced the symbol “ \approx ” with equality. We note that the expression inside the curly brackets in eq. (A.15) behaves at small \bar{r} as \bar{r}^3 , and $\Delta\tilde{\Pi}(P)$ consequently indeed vanishes at zero temperature, and behaves as $1/P^2$ at large P .

A.1.3 $\bar{\Pi}(P)$

Finally, we divide the function $\bar{\Pi}(P)$ into two parts by separating from it the contribution of the Matsubara zero-mode,

$$\begin{aligned} \bar{\Pi}(P) &\equiv \not\int_Q \frac{1}{Q^4(Q-P)^2} = \not\int'_Q \frac{1}{Q^4(Q-P)^2} + T\Lambda^{2\epsilon} \int \frac{d^{3-2\epsilon}\mathbf{q}}{(2\pi)^{3-2\epsilon}} \frac{1}{q^4[(\mathbf{q}-\mathbf{p})^2 + p_0^2]} \\ &\equiv \bar{\Pi}_r(P) + \bar{\Pi}_0(P), \end{aligned} \tag{A.16}$$

where the prime in the upper right corner of the sum-integral symbol signifies the leaving out of the zero-mode $q_0 = 0$ from the corresponding Matsubara sum. Both of these parts are UV-finite. In three dimensions, one obtains the representations

$$\bar{\Pi}_r(P) = \frac{T}{2(4\pi)^2} \int d^3\mathbf{r} \frac{1}{r} e^{i\mathbf{p}\cdot\mathbf{r}} \sum_{q_0 \neq 0} \frac{1}{|q_0|} e^{-(|q_0|+|q_0-p_0|)r}, \tag{A.17}$$

$$\begin{aligned} \bar{\Pi}_0(P) &= -\frac{T}{(2\pi)^2} \int_0^\infty dq \frac{1}{q^2} \left\{ \frac{2}{P^2} + \frac{1}{2pq} \ln \left[\frac{(p-q)^2 + p_0^2}{(p+q)^2 + p_0^2} \right] \right\} \\ &= -\frac{T}{4\pi} \frac{|p_0|}{P^4} = -\frac{T}{2(4\pi)^2} \int d^3\mathbf{r} e^{i\mathbf{p}\cdot\mathbf{r}} e^{-|p_0|r}, \end{aligned} \tag{A.18}$$

where in the latter case we have first (in $d = 3 - 2\epsilon$ dimensions) subtracted from the integrand the term $1/q^4 P^2$, the integral of which vanishes in dimensional regularization but which renders the expression for $\bar{\Pi}_0(P)$ IR convergent.

A.2 Strategy for determining S_1

As often in the evaluation of multi-loop (sum-)integrals, the first and in some sense also the most important step is to divide the integrand into two types of terms: ones that are divergent, but sufficiently simple to allow for an analytic evaluation, and others that are perhaps complicated but both IR and UV convergent. In the case of the sum-integral S_1 defined in eq. (2.21), we do this by first decomposing $\Pi(P)$ according to eq. (A.1), and then interchanging the order of sum-integrals, $P \leftrightarrow Q$, in the terms involving $\Pi^{(0)}(P)$ and $2\mathcal{I}_1/P^2$. This leads to the following decomposition of the original sum-integral:

$$\begin{aligned} S_1 &= \not\int'_P \Delta\Pi(P)\bar{\Pi}_r(P) + \not\int'_P \Delta\Pi(P)\bar{\Pi}_0(P) + 2\mathcal{I}_1 \not\int'_P \frac{\Pi(P)}{P^4} + \not\int'_P \frac{\tilde{\Pi}(P)}{P^4} \\ &\equiv S_1^I + S_1^{II} + S_1^{III} + S_1^{IV}. \end{aligned} \tag{A.19}$$

In S_1^{IV} we repeat the split-up procedure by using eq. (A.7) for $\tilde{\Pi}(P)$. Furthermore, we take care of IR divergences by separating the contribution of the zero-mode from the Matsubara sums where necessary:

$$\begin{aligned} S_1^{II} &= \not\int'_P \Delta\Pi(P)\bar{\Pi}_0(P) + T\Lambda^{2\epsilon} \int \frac{d^{3-2\epsilon}\mathbf{p}}{(2\pi)^{3-2\epsilon}} \Pi(p_0 = 0, p)\bar{\Pi}_0(p_0 = 0, p) \\ &\equiv S_1^{II,a} + S_1^{II,b}, \end{aligned} \tag{A.20}$$

$$\begin{aligned}
 S_1^{\text{III}} &= 2\mathcal{I}_1 \left\{ \not\int'_P \frac{\Pi^{(T)}(P)}{P^4} + T\Lambda^{2\epsilon} \int \frac{d^{3-2\epsilon}\mathbf{p}}{(2\pi)^{3-2\epsilon}} \frac{\Pi(p_0=0, p)}{p^4} + \not\int'_P \frac{\Pi^{(0)}(P)}{P^4} \right\} \\
 &\equiv S_1^{\text{III,a}} + S_1^{\text{III,b}} + S_1^{\text{III,c}} , \tag{A.21}
 \end{aligned}$$

$$\begin{aligned}
 S_1^{\text{IV}} &= \not\int'_P \frac{\Delta\tilde{\Pi}(P)}{P^4} + T\Lambda^{2\epsilon} \int \frac{d^{3-2\epsilon}\mathbf{p}}{(2\pi)^{3-2\epsilon}} \frac{\tilde{\Pi}(p_0=0, p)}{p^4} + \beta\mathcal{I}_1 \not\int'_P \frac{1}{(P^2)^{2+\epsilon}} + \not\int'_P \frac{\tilde{\Pi}^{(0)}(P)}{P^4} \\
 &\equiv S_1^{\text{IV,a}} + S_1^{\text{IV,b}} + S_1^{\text{IV,c}} + S_1^{\text{IV,d}} . \tag{A.22}
 \end{aligned}$$

The only further manipulations performed here are the dropping of terms that vanish in dimensional regularization; this happens in Matsubara zero-mode integrals without scales, after writing $\Delta\Pi \rightarrow \Pi - \Pi^{(0)} - 2\mathcal{I}_1/P^2$ in $S_1^{\text{II,b}}$, $\Pi^{(T)} \rightarrow \Pi - \Pi^{(0)}$ in $S_1^{\text{III,b}}$, and $\Delta\tilde{\Pi} \rightarrow \tilde{\Pi} - \tilde{\Pi}^{(0)} - \mathcal{I}_1\Pi^{(0)}$ in $S_1^{\text{IV,b}}$.

We have now separated the original sum-integral into ten pieces, which fall into the following sub-categories:

- Finite terms that can be evaluated numerically: $S_1^{\text{I}}, S_1^{\text{II,a}}, S_1^{\text{III,a}}, S_1^{\text{IV,a}}$.
- Possibly divergent terms that can be evaluated analytically through the introduction of Feynman parameters: $S_1^{\text{II,b}}, S_1^{\text{III,b}}, S_1^{\text{IV,b}}$.
- Terms that are trivial to compute due to eq. (2.19): $S_1^{\text{III,c}}, S_1^{\text{IV,c}}, S_1^{\text{IV,d}}$.

Below, we will go through the evaluation of the sum-integrals of the first category in detail and outline the calculation of those in the second category.

A.2.1 S_1^{I}

It is straightforward to see that S_1^{I} is finite in three dimensions, so we will immediately set $\epsilon = 0$. Expressing $\Delta\Pi(P)$ and $\tilde{\Pi}_r(P)$ in terms of their Fourier representations, given in eqs. (A.3) and (A.17), enables us to perform the \mathbf{p} -integral in the definition of S_1^{I} to give a δ -function in coordinate space. This leaves us with the result

$$\begin{aligned}
 S_1^{\text{I}} &= \frac{T^3}{2(4\pi)^4} \int d^3\mathbf{r} \frac{1}{r^3} \left(\coth \bar{r} - \frac{1}{\bar{r}} - \frac{\bar{r}}{3} \right) \sum_{p_0} \sum_{q_0 \neq 0} \frac{e^{-(|p_0|+|q_0|+|q_0-p_0|)r}}{|q_0|} \\
 &= \frac{2T^2}{(4\pi)^4} \int_0^\infty dr \frac{1}{r} \left(\coth r - \frac{1}{r} - \frac{r}{3} \right) \left\{ \frac{1}{e^{2r}-1} - \coth r \ln(1 - e^{-2r}) \right\} \\
 &\approx -\frac{T^2}{(4\pi)^4} \times 0.0269726622737(1) , \tag{A.23}
 \end{aligned}$$

where we have analytically performed the sums over p_0 and q_0 (in this order) and later dropped the bars over the dimensionless coordinate variable. The last one-dimensional integral is of a form that might allow for an analytic evaluation, but for the purposes of this paper we have simply computed its value numerically. The number in parentheses estimates the uncertainty of the last digit.

A.2.2 $S_1^{\text{II,a}}$

For $S_1^{\text{II,a}}$, we proceed along the lines of the previous section, employing this time the Fourier representation of $\bar{\Pi}_0(P)$ given in eq. (A.18). After scaling all variables dimensionless, we obtain

$$\begin{aligned} S_1^{\text{II,a}} &= -\frac{T^2}{(4\pi)^5} \int d^3\mathbf{r} \frac{1}{r^2} \left(\coth r - \frac{1}{r} - \frac{r}{3} \right) \sum_{n \neq 0} e^{-2|n|r} \\ &= -\frac{2T^2}{(4\pi)^4} \int_0^\infty dr \left(\coth r - \frac{1}{r} - \frac{r}{3} \right) \frac{1}{e^{2r} - 1} \\ &\approx \frac{T^2}{(4\pi)^4} \times 0.0134942763002(1), \end{aligned} \tag{A.24}$$

where the last integration was performed numerically.²

A.2.3 $S_1^{\text{III,a}}$

Expressing $1/P^4$ and $\Pi^{(T)}(P)$ in terms of their respective Fourier integrals, we again perform the three-dimensional momentum integral and end up with one coordinate space integral and an infinite Matsubara sum. In dimensionless variables, we get

$$\begin{aligned} S_1^{\text{III,a}} &= \frac{4\mathcal{I}_1}{(4\pi)^5} \int d^3\mathbf{r} \frac{1}{r^2} \left(\coth r - \frac{1}{r} \right) \sum_{n \neq 0} \frac{1}{|n|} e^{-2|n|r} \\ &= -\frac{2T^2}{3(4\pi)^4} \int_0^\infty dr \left(\coth r - \frac{1}{r} \right) \ln(1 - e^{-2r}) \\ &\approx \frac{T^2}{(4\pi)^4} \times 0.0625154109468(1), \end{aligned} \tag{A.25}$$

where the last integration was performed numerically.³

A.2.4 $S_1^{\text{IV,a}}$

To obtain $S_1^{\text{IV,a}}$, we repeat the calculation of $S_1^{\text{III,a}}$, but simply replace $\Pi^{(T)}(P)$ by $\Delta\tilde{\Pi}(P)$ and accordingly eq. (A.3) [without $-\bar{r}/3$] by eq. (A.15). This gives

$$\begin{aligned} S_1^{\text{IV,a}} &= \frac{T^2}{2(4\pi)^5} \int d^3\mathbf{r} \frac{1}{r^4} \sum_{n \neq 0} \frac{1}{|n|} \left\{ r \operatorname{csch}^2 r + (2 + |n|r)(|n| + \coth r) - \right. \\ &\quad \left. - \frac{1}{r} (3 + 3|n|r + n^2 r^2) - \frac{r}{3} (1 + |n|r) \right\} e^{-2|n|r} \\ &= \frac{T^2}{(4\pi)^4} \int_0^\infty dr \frac{1}{r^2} \left\{ \frac{1}{e^{2r} - 1} \left(r \coth r - 1 - \frac{r^2}{3} \right) + \right. \\ &\quad \left. + \frac{1}{r} \ln(1 - e^{-2r}) \left(3 + \frac{r^2}{3} - r^2 \operatorname{csch}^2 r - 2r \coth r \right) \right\} \\ &\approx -\frac{T^2}{(4\pi)^4} \times 0.0004627085472(1). \end{aligned} \tag{A.26}$$

²Its analytic value is $T^2[1 + \gamma_E + \pi^2/36 - \ln(2\pi)]/(4\pi)^4$.

³Its analytic value is $T^2(\gamma_E^2 + 2\gamma_1)/3(4\pi)^4$, where γ_1 refers to the first Stieltjes gamma constant (cf. the explanation following eq. (3.12)).

A.2.5 $S_1^{\text{II,b}}$, $S_1^{\text{III,b}}$ and $S_1^{\text{IV,b}}$

The sum-integrals $S_1^{\text{II,b}}$, $S_1^{\text{III,b}}$ and $S_1^{\text{IV,b}}$ have one thing in common: the “outer” sum-integral is restricted to the Matsubara zero-mode $p_0 = 0$, while the “inner” loop contains only one Matsubara sum. This implies that by introducing Feynman parameters, it is always possible to perform all of the momentum integrals analytically, leaving in the end a simple sum of a form that produces just a Riemann ζ -function. As a first step, we quote the following results for the “static limits” ($p_0 = 0$) of the various functions, obtained through Feynman parametrization:

$$\Pi(p_0 = 0, p) = T\Lambda^{2\epsilon} \frac{\Gamma(1/2 + \epsilon)}{(4\pi)^{3/2-\epsilon}} \sum_{q_0} \int_0^1 dx \frac{1}{[x(1-x)p^2 + q_0^2]^{1/2+\epsilon}}, \quad (\text{A.27})$$

$$\bar{\Pi}_0(p_0 = 0, p) = T\Lambda^{2\epsilon} \frac{2^{1+2\epsilon} \sqrt{\pi}}{(4\pi)^{3/2-\epsilon}} \frac{\Gamma(3/2 + \epsilon)\Gamma(-1/2 - \epsilon)}{\Gamma(-\epsilon)} \frac{1}{(p^2)^{3/2+\epsilon}}, \quad (\text{A.28})$$

$$\tilde{\Pi}(p_0 = 0, p) = \frac{\beta T\Lambda^{2\epsilon}}{(4\pi)^{3/2-\epsilon}} \frac{\Gamma(-1/2 + 2\epsilon)}{\Gamma(\epsilon)} \sum_{q_0} \int_0^1 dx \frac{(1-x)^{-1+\epsilon}}{[x(1-x)p^2 + q_0^2]^{-1/2+2\epsilon}}, \quad (\text{A.29})$$

which we then substitute into the definitions of the sum-integrals in question.

For $S_1^{\text{II,b}}$, we obtain in the fashion described above

$$\begin{aligned} S_1^{\text{II,b}} &= T^3 \Lambda^{6\epsilon} \frac{2^{1+2\epsilon} \sqrt{\pi}}{(4\pi)^{3-2\epsilon}} \frac{\Gamma(3/2 + \epsilon)\Gamma(1/2 + \epsilon)\Gamma(-1/2 - \epsilon)}{\Gamma(-\epsilon)} \times \\ &\quad \times \sum_{q_0} \int_0^1 dx \int \frac{d^{3-2\epsilon} \mathbf{p}}{(2\pi)^{3-2\epsilon}} \frac{1}{(p^2)^{3/2+\epsilon} [x(1-x)p^2 + q_0^2]^{1/2+\epsilon}} \\ &= T^3 \Lambda^{6\epsilon} \frac{2^{1+2\epsilon} \sqrt{\pi}}{(4\pi)^{3-2\epsilon}} \frac{\Gamma(3/2 + \epsilon)\Gamma(1/2 + \epsilon)\Gamma(-1/2 - \epsilon)}{\Gamma(-\epsilon)} \left\{ \sum_{q_0} (q_0^2)^{-1/2-3\epsilon} \right\} \times \\ &\quad \times \left\{ \int_0^1 dx [x(1-x)]^{2\epsilon} \right\} \left\{ \frac{2\pi^{3/2-\epsilon}}{\Gamma(3/2 - \epsilon)} \frac{1}{(2\pi)^{3-2\epsilon}} \int_0^\infty dp \frac{p^{-1-4\epsilon}}{(p^2 + 1)^{1/2+\epsilon}} \right\}, \quad (\text{A.30}) \end{aligned}$$

where in the last form we have through rescalings of integration variables reduced the sum-integral into a product of sums and integrals that can each be trivially evaluated analytically. Performing this task, the final result for the function reads

$$S_1^{\text{II,b}} = -\frac{T^2}{6(4\pi)^4} \left(\frac{1}{\epsilon} + 3 \ln \frac{\Lambda^2}{4\pi T^2} - 2 + 3\gamma_E \right) + \mathcal{O}(\epsilon). \quad (\text{A.31})$$

To evaluate $S_1^{\text{III,b}}$, we note that a calculation exactly parallel to the above produces

$$\begin{aligned} S_1^{\text{III,b}} &= 2\mathcal{I}_1 T^2 \Lambda^{4\epsilon} \frac{\Gamma(1/2 + \epsilon)}{(4\pi)^{3/2-\epsilon}} \sum_{q_0} \int_0^1 dx \int \frac{d^{3-2\epsilon} \mathbf{p}}{(2\pi)^{3-2\epsilon}} \frac{1}{p^4 [x(1-x)p^2 + q_0^2]^{1/2+\epsilon}} \\ &= 2\mathcal{I}_1 T^2 \Lambda^{4\epsilon} \frac{\Gamma(1/2 + \epsilon)}{(4\pi)^{3/2-\epsilon}} \left\{ \sum_{q_0} (q_0^2)^{-1-2\epsilon} \right\} \left\{ \int_0^1 dx [x(1-x)]^{1/2+\epsilon} \right\} \times \\ &\quad \times \left\{ \frac{2\pi^{3/2-\epsilon}}{\Gamma(3/2 - \epsilon)} \frac{1}{(2\pi)^{3-2\epsilon}} \int_0^\infty dp \frac{p^{-2-2\epsilon}}{(p^2 + 1)^{1/2+\epsilon}} \right\} \\ &= -\frac{T^2}{(4\pi)^4} \frac{\pi^2}{36} + \mathcal{O}(\epsilon). \quad (\text{A.32}) \end{aligned}$$

For $S_1^{\text{IV,b}}$, we finally get

$$\begin{aligned}
 S_1^{\text{IV,b}} &= \frac{\beta T^2 \Lambda^{4\epsilon}}{(4\pi)^{3/2-\epsilon}} \frac{\Gamma(-1/2+2\epsilon)}{\Gamma(\epsilon)} \sum_{q_0} \int_0^1 dx \int \frac{d^{3-2\epsilon} \mathbf{p}}{(2\pi)^{3-2\epsilon}} \frac{(1-x)^{-1+\epsilon}}{p^4 [x(1-x)p^2 + q_0^2]^{-1/2+2\epsilon}} \\
 &= \frac{\beta T^2 \Lambda^{4\epsilon}}{(4\pi)^{3/2-\epsilon}} \frac{\Gamma(-1/2+2\epsilon)}{\Gamma(\epsilon)} \left\{ \sum_{q_0} (q_0^2)^{-3\epsilon} \right\} \left\{ \int_0^1 dx x^{1/2+\epsilon} (1-x)^{-1/2+2\epsilon} \right\} \times \\
 &\quad \times \left\{ \frac{2\pi^{3/2-\epsilon}}{\Gamma(3/2-\epsilon)} \frac{1}{(2\pi)^{3-2\epsilon}} \int_0^\infty dp \frac{p^{-2-2\epsilon}}{(p^2+1)^{-1/2+2\epsilon}} \right\} \\
 &= \frac{T^2}{6(4\pi)^4} \left[\frac{1}{\epsilon} + 3 \ln \frac{\Lambda^2}{4\pi T^2} + 1 - 3\gamma_E + 6 \ln(2\pi) \right] + \mathcal{O}(\epsilon). \tag{A.33}
 \end{aligned}$$

A.2.6 $S_1^{\text{III,c}}$, $S_1^{\text{IV,c}}$ and $S_1^{\text{IV,d}}$

The last three sum-integrals are trivial to evaluate, as their analytic values can be obtained by a straightforward application of eq. (2.19). The sum of the three gives

$$\begin{aligned}
 S_1^{\text{III,c}} + S_1^{\text{IV,c}} + S_1^{\text{IV,d}} &= 3\beta \mathcal{I}_1 \mathcal{I}_{2+\epsilon} + \tilde{\beta} \tilde{\mathcal{I}}_{1+2\epsilon} \\
 &= \frac{T^2}{8(4\pi)^4} \left\{ \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \left[3 \ln \frac{\Lambda^2}{4\pi T^2} + \frac{17}{6} + \gamma_E + 2 \frac{\zeta'(-1)}{\zeta(-1)} \right] + \right. \\
 &\quad + \frac{9}{2} \left(\ln \frac{\Lambda^2}{4\pi T^2} \right)^2 + \left(\frac{17}{2} + 3\gamma_E + 6 \frac{\zeta'(-1)}{\zeta(-1)} \right) \ln \frac{\Lambda^2}{4\pi T^2} + \frac{83}{12} + \\
 &\quad \left. + \frac{13\pi^2}{12} + \frac{7\gamma_E}{2} - \frac{15\gamma_E^2}{2} + (5 + 2\gamma_E) \frac{\zeta'(-1)}{\zeta(-1)} + 2 \frac{\zeta''(-1)}{\zeta(-1)} - 16\gamma_1 \right\} \\
 &\quad + \mathcal{O}(\epsilon), \tag{A.34}
 \end{aligned}$$

where γ_1 refers to the first Stieltjes gamma constant (cf. the explanation following eq. (3.12)).

A.2.7 The full result for S_1

Collecting all the parts above, the final result for the sum-integral S_1 is seen to read

$$\begin{aligned}
 S_1 &= \frac{T^2}{8(4\pi)^4} \left\{ \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \left[3 \ln \frac{\Lambda^2}{4\pi T^2} + \frac{17}{6} + \gamma_E + 2 \frac{\zeta'(-1)}{\zeta(-1)} \right] + \frac{9}{2} \left(\ln \frac{\Lambda^2}{4\pi T^2} \right)^2 + \right. \\
 &\quad + \left(\frac{17}{2} + 3\gamma_E + 6 \frac{\zeta'(-1)}{\zeta(-1)} \right) \ln \frac{\Lambda^2}{4\pi T^2} + \frac{131}{12} + \frac{31\pi^2}{36} + 8 \ln(2\pi) - \frac{9\gamma_E}{2} - \\
 &\quad \left. - \frac{15\gamma_E^2}{2} + (5 + 2\gamma_E) \frac{\zeta'(-1)}{\zeta(-1)} + 2 \frac{\zeta''(-1)}{\zeta(-1)} - 16\gamma_1 + 0.388594531408(4) + \mathcal{O}(\epsilon) \right\}. \tag{A.35}
 \end{aligned}$$

Evaluating the non-logarithmic terms numerically leads to the result in eq. (2.23).

A.3 Strategy for determining S_2

As with S_1 , we begin the evaluation of our only genuinely 4-loop sum-integral S_2 , defined in eq. (2.22), by dividing it into several pieces that are then calculated separately. Defining

a regularized, or subtracted, zero-temperature limit for the function $\Pi(P)$ by

$$\begin{aligned}
 \Pi_s^{(0)}(P) &\equiv \Pi^{(0)}(P) - \frac{1}{(4\pi)^2\epsilon} \\
 &= -\frac{1}{(4\pi)^2} \left\{ \ln \frac{P^2}{(2\pi T)^2} - \ln \frac{\Lambda^2}{4\pi T^2} - 2 - 2 \ln 2 + \gamma_E \right\} + \mathcal{O}(\epsilon) \\
 &\equiv -\frac{1}{(4\pi)^2} \left\{ \ln \frac{P^2}{(2\pi T)^2} + \kappa \right\} + \mathcal{O}(\epsilon), \tag{A.36}
 \end{aligned}$$

we write $\Pi(P) = 1/(4\pi)^2\epsilon + \Pi_s^{(0)}(P) + \Pi^{(T)}(P)$, and drop terms that vanish in dimensional regularization. This leads to

$$\begin{aligned}
 S_2 &= \not\int'_P \left[\Pi^{(T)}(P) \right]^3 + 3 \not\int'_P \left[\Pi^{(T)}(P) \right]^2 \Pi_s^{(0)}(P) + 3 \not\int'_P \Pi^{(T)}(P) \left[\Pi_s^{(0)}(P) \right]^2 + \\
 &\quad + \not\int'_P \left[\Pi_s^{(0)}(P) \right]^3 - \frac{3}{(4\pi)^4\epsilon^2} \not\int'_P \Pi(P) \\
 &\equiv S_2^{\text{I}} + S_2^{\text{II}} + S_2^{\text{III}} + S_2^{\text{IV}} + S_2^{\text{V}}. \tag{A.37}
 \end{aligned}$$

Here, we again decompose the various sum-integrals into further pieces according to

$$\begin{aligned}
 S_2^{\text{I}} &= \not\int'_P \left[\Pi^{(T)}(P) \right]^3 + T\Lambda^{2\epsilon} \int \frac{d^{3-2\epsilon}\mathbf{p}}{(2\pi)^{3-2\epsilon}} \left[\Pi^{(T)}(p_0 = 0, p) \right]^3 \\
 &\equiv S_2^{\text{I,a}} + S_2^{\text{I,b}}, \tag{A.38}
 \end{aligned}$$

$$\begin{aligned}
 S_2^{\text{II}} &= 3 \not\int'_P \left\{ \left[\Pi^{(T)}(P) \right]^2 - \frac{4(\mathcal{I}_1)^2}{P^4} \right\} \Pi_s^{(0)}(P) + 12(\mathcal{I}_1)^2 \not\int'_P \frac{\Pi_s^{(0)}(P)}{P^4} \\
 &\equiv S_2^{\text{II,a}} + S_2^{\text{II,b}}, \tag{A.39}
 \end{aligned}$$

$$\begin{aligned}
 S_2^{\text{III}} &= 3 \not\int'_P \Delta' \Pi(P) \left[\Pi_s^{(0)}(P) \right]^2 + 6\mathcal{I}_1 \not\int'_P \frac{\left[\Pi_s^{(0)}(P) \right]^2}{P^2} + \\
 &\quad + \frac{24T^4 J_1}{3-2\epsilon} \not\int'_P \left[\Pi_s^{(0)}(P) \right]^2 \left[\frac{1}{P^4} - (4-2\epsilon) \frac{p_0^2}{P^6} \right] \\
 &\equiv S_2^{\text{III,a}} + S_2^{\text{III,b}} + S_2^{\text{III,c}}. \tag{A.40}
 \end{aligned}$$

This time we classify the various parts as

- Finite (but complicated) terms that can be evaluated numerically: $S_2^{\text{I,a}}$, $S_2^{\text{II,a}}$, $S_2^{\text{III,a}}$.
- Possibly divergent terms, whose divergent parts can be evaluated analytically and finite parts numerically: $S_2^{\text{I,b}}$.
- Terms that are trivial to compute due to eq. (2.19): $S_2^{\text{II,b}}$, $S_2^{\text{III,b}}$, $S_2^{\text{III,c}}$, S_2^{IV} , S_2^{V} .

We move on to present the evaluation of the first four sum-integrals in detail.

A.3.1 $S_2^{\text{I,a}}$

The term $S_2^{\text{I,a}}$ is entirely finite in three dimensions, so we set $\epsilon = 0$ and proceed to its numerical evaluation. There are two comparably simple ways of doing this, as one can either work entirely in coordinate space, or use the relation

$$\Pi^{(T)}(P) = \frac{T}{4\pi p} \int_0^\infty dr \frac{\sin pr}{r} \left(\coth \bar{r} - \frac{1}{\bar{r}} \right) e^{-|p_0|r}, \quad (\text{A.41})$$

and perform the sum-integration over P directly in momentum space. Here, we choose the latter approach, but have verified the result also by the former method. Scaling all variables dimensionless, we straightforwardly get

$$\begin{aligned} S_2^{\text{I,a}} &= \frac{T^4}{(4\pi)^4} \frac{4}{\pi} \sum_{n=1}^\infty \int_0^\infty dp \frac{1}{p} \left\{ \int_0^\infty dr \frac{\sin(pr)}{r} \left(\coth r - \frac{1}{r} \right) e^{-nr} \right\}^3 \\ &\approx \frac{T^4}{(4\pi)^4} \times 0.0092313322549(1). \end{aligned} \quad (\text{A.42})$$

The summation can be accelerated by noting that for large n , the integral inside the curly brackets yields $p/3(p^2+n^2)$, and the subsequent p -integral $\pi/432n^3$. This leading behaviour can be subtracted and the corresponding sum carried out analytically.

A.3.2 $S_2^{\text{II,a}}$

The only potential divergence in $S_2^{\text{II,a}}$ is of an IR type, and produced by the zero-mode piece $p_0 = 0$ of the $1/P^4$ subtraction term. However, there is in fact no divergence in dimensional regularization, given that the zero-mode piece does not contribute: $\int d^{3-2\epsilon} \mathbf{p} [\Pi^{(0)}(p_0 = 0, p) - 1/(4\pi)^2 \epsilon]/p^4 = 0$. We may therefore set $\epsilon = 0$ from the beginning, if we agree to throw away the subtraction term for $p_0 = 0$. Using furthermore the result of eqs. (2.32) and (2.34) of ref. [1],

$$\not\int_P \left\{ \left[\Pi^{(T)}(P) \right]^2 - \frac{4(\mathcal{I}_1)^2}{P^4} \right\} = \frac{1}{(4\pi)^2} \left(\frac{T^2}{12} \right)^2 \left[\frac{28}{15} - 8\gamma_E + 24 \frac{\zeta'(-1)}{\zeta(-1)} - 16 \frac{\zeta'(-3)}{\zeta(-3)} \right], \quad (\text{A.43})$$

as well as the representations for $\Pi_s^{(0)}$ and $\Pi^{(T)}$ as in eqs. (A.36), (A.41), we thus obtain

$$\begin{aligned} S_2^{\text{II,a}} &= -\frac{T^4}{(4\pi)^4} \frac{3}{\pi} \sum_n \int_0^\infty dp \ln(p^2 + n^2) \times \\ &\quad \times \left\{ \left[\int_0^\infty dr \frac{\sin(pr)}{r} \left(\coth r - \frac{1}{r} \right) e^{-|n|r} \right]^2 - \frac{p^2}{9} \frac{1 - \delta_n}{(p^2 + n^2)^2} \right\} - \\ &\quad - \frac{3\kappa}{(4\pi)^4} \left(\frac{T^2}{12} \right)^2 \left[\frac{28}{15} - 8\gamma_E + 24 \frac{\zeta'(-1)}{\zeta(-1)} - 16 \frac{\zeta'(-3)}{\zeta(-3)} \right]. \end{aligned} \quad (\text{A.44})$$

For large n , the expression inside the curly brackets approaches $4p^2(p^2 - 3n^2)/135(p^2 + n^2)^4$, and the subsequent p -integral yields $\pi[7 - 12 \ln(2n)]/3240n^3$. This term can again be subtracted and the corresponding sums carried out analytically, to accelerate the convergence

of the remaining numerical summation. We thus get

$$\begin{aligned}
 S_2^{\text{II,a}} &= \frac{T^4}{(4\pi)^4} \left\{ \frac{1}{48} \left(\ln \frac{\Lambda^2}{4\pi T^2} + 2 + 2 \ln 2 - \gamma_E \right) \left[\frac{28}{15} - 8\gamma_E + 24 \frac{\zeta'(-1)}{\zeta(-1)} - 16 \frac{\zeta'(-3)}{\zeta(-3)} \right] \right. \\
 &\quad \left. + 2.0344721052(1) \right\} \\
 &= \frac{T^4}{(4\pi)^4} \left\{ \left[\frac{7}{180} - \frac{\gamma_E}{6} + \frac{1}{2} \frac{\zeta'(-1)}{\zeta(-1)} - \frac{1}{3} \frac{\zeta'(-3)}{\zeta(-3)} \right] \ln \frac{\Lambda^2}{4\pi T^2} + 4.0572056435(1) \right\}.
 \end{aligned} \tag{A.45}$$

A.3.3 $S_2^{\text{III,a}}$

With $S_2^{\text{III,a}}$, we begin the calculation by noting that if we write

$$\left(\Pi_s^{(0)}(P) \right)^2 = \frac{1}{(4\pi)^4} \left\{ \left(\ln \frac{P^2}{(2\pi T)^2} \right)^2 - \kappa^2 \right\} - \frac{2\kappa}{(4\pi)^2} \Pi_s^{(0)}(P), \tag{A.46}$$

and use the result in eq. (D.14) of ref. [1],

$$\oint_P \Delta' \Pi(P) \Pi_s^{(0)}(P) = -\frac{1}{(4\pi)^2} \left(\frac{T^2}{12} \right)^2 \left\{ \frac{46}{15} - \frac{12}{5} \gamma_E + 4 \frac{\zeta'(-1)}{\zeta(-1)} - \frac{8}{5} \frac{\zeta'(-3)}{\zeta(-3)} \right\}, \tag{A.47}$$

as well as that in eq. (2.19) of our section 2, in order to show that $\oint_P \Delta' \Pi(P) = \oint_P [\Pi(P) - \Pi^{(0)}(P) - \dots] = \mathcal{O}(\epsilon)$, so that the κ^2 -term can be neglected, we can immediately reduce the evaluation of the original sum-integral into that of the simpler function

$$\oint_P \Delta' \Pi(P) \left(\ln \frac{P^2}{(2\pi T)^2} \right)^2. \tag{A.48}$$

In performing the latter task, we again have a choice between working entirely in coordinate space or combining (spatial) coordinate and momentum space techniques; this time we for illustration choose the former approach.

We begin the calculation by recalling the result of eq. (A.9), from which we obtain by differentiation with respect to α

$$\begin{aligned}
 \int \frac{d^3 \mathbf{p}}{(2\pi)^3} e^{-i\mathbf{p}\cdot\mathbf{r}} [\ln(p^2 + m^2)]^2 &= -\frac{e^{-|m|r}}{\pi r^3} \left\{ 2 - (1 + |m|r) \left(\gamma_E - \ln \frac{2|m|}{r} \right) \right\} - \\
 &\quad - \frac{e^{|m|r}}{\pi r^3} (1 - |m|r) \text{E}_1(2|m|r) \\
 &\equiv \frac{e^{-|m|r}}{\pi r^3} f(r, m).
 \end{aligned} \tag{A.49}$$

Here, we have used the relation

$$\left[\frac{\partial}{\partial \nu} K_\nu(z) \right]_{\nu=3/2} = \sqrt{\frac{\pi}{2z^3}} e^{-z} \left\{ 2 - (z-1) e^{2z} \text{E}_1(2z) \right\}, \tag{A.50}$$

in which the exponential integral E_1 is related to the exponential integral function Ei through $E_1(z) = -Ei(-z)$. We also note that for $m = 0$, the function $f(r, m)$ becomes

$$f(r, 0) = 2(\ln r - 1 + \gamma_E). \quad (\text{A.51})$$

Going now to coordinate space in the sum-integral and scaling the integration variables dimensionless, we obtain

$$\begin{aligned} \not\int_P \Delta' \Pi(P) \left(\ln \frac{P^2}{(2\pi T)^2} \right)^2 &= T^4 \sum_n \int_0^\infty dr \frac{1}{r^3} f(r, n) \times \\ &\quad \times \left(\coth r - \frac{1}{r} - \frac{r}{3} + (1 - \delta_n) \frac{r^3}{45} \right) e^{-2|n|r} \\ &= 2T^4 \left\{ \int_0^\infty dr \frac{1}{r^3} (\ln r - 1 + \gamma_E) \left(\coth r - \frac{1}{r} - \frac{r}{3} \right) + \right. \\ &\quad \left. + \sum_{n=1}^\infty \int_0^\infty dr \frac{1}{r^3} f(r, n) \left(\coth r - \frac{1}{r} - \frac{r}{3} + \frac{r^3}{45} \right) e^{-2nr} \right\}, \end{aligned} \quad (\text{A.52})$$

where all terms are both IR and UV convergent thanks to the $(1 - \delta_n)$ factor above. Given that it is perfectly finite, the zero-mode integral is furthermore simple to evaluate through a procedure introduced in ref. [1]: one temporarily introduces a convergence factor r^α into the integrand, writes $\coth r = 1 + 2/(e^{2r} - 1)$, throws out all terms that are simple powers of r , performs the remaining integrals and finally proceeds to the limit $\alpha \rightarrow 0$ (which result can also be reproduced numerically). A straightforward calculation employing numerical integration for the non-zero mode terms then yields

$$\begin{aligned} \not\int_P \Delta' \Pi(P) \left(\ln \frac{P^2}{(2\pi T)^2} \right)^2 &= 2T^4 \left\{ \frac{1}{\pi^2} [(1 - \gamma_E - \ln \pi)\zeta(3) + \zeta'(3)] + \right. \\ &\quad \left. + \sum_{n=1}^\infty \int_0^\infty dr \frac{1}{r^3} f(r, n) \left(\coth r - \frac{1}{r} - \frac{r}{3} + \frac{r^3}{45} \right) e^{-2nr} \right\} \\ &= T^4 \left\{ \frac{2}{\pi^2} [(1 - \gamma_E - \ln \pi)\zeta(3) + \zeta'(3)] - 0.002580375031(1) \right\} \\ &\approx -T^4 \times 0.218586170715(1). \end{aligned} \quad (\text{A.53})$$

Like before, the large- n behaviour can be worked out analytically and subtracted, to accelerate the convergence of the summation: it is $2T^4[11 - 24 \ln(2n)]/9072n^3$.

Collecting everything together, we finally have

$$\begin{aligned} S_2^{\text{III,a}} &= -\frac{T^4}{(4\pi)^4} \left\{ \frac{1}{24} \left(\ln \frac{\Lambda^2}{4\pi T^2} + 2 + 2 \ln 2 - \gamma_E \right) \left[\frac{46}{15} - \frac{12}{5} \gamma_E + 4 \frac{\zeta'(-1)}{\zeta(-1)} - \frac{8}{5} \frac{\zeta'(-3)}{\zeta(-3)} \right] \right. \\ &\quad \left. - \frac{6}{\pi^2} [(1 - \gamma_E - \ln \pi)\zeta(3) + \zeta'(3)] + 0.007741125093(3) \right\} \quad (\text{A.54}) \\ &= -\frac{T^4}{(4\pi)^4} \left\{ \left[\frac{23}{180} - \frac{\gamma_E}{10} + \frac{1}{6} \frac{\zeta'(-1)}{\zeta(-1)} - \frac{1}{15} \frac{\zeta'(-3)}{\zeta(-3)} \right] \ln \frac{\Lambda^2}{4\pi T^2} + 1.661043190056(3) \right\}. \end{aligned}$$

A.3.4 $S_2^{\text{I,b}}$

The function $S_2^{\text{I,b}}$ has one unique property: while entirely UV finite, it is the only sum-integral encountered in the present calculation that has a non-trivial logarithmic IR divergence that does not vanish in dimensional regularization. To see this, we note that the contribution of the Matsubara zero-mode to $\Pi^{(T)}(p_0 = 0, p)$ [the superscript (T) is irrelevant here] reads

$$\begin{aligned} \Pi_{\text{IR}}^{(T)}(p_0 = 0, p) &\equiv T\Lambda^{2\epsilon} \int \frac{d^{3-2\epsilon}\mathbf{q}}{(2\pi)^{3-2\epsilon}} \frac{1}{q^2(\mathbf{q} - \mathbf{p})^2} \\ &= T\Lambda^{2\epsilon} \frac{2^{2\epsilon}\sqrt{\pi}}{(4\pi)^{3/2-\epsilon}} \frac{\Gamma(1/2 + \epsilon)\Gamma(1/2 - \epsilon)}{\Gamma(1 - \epsilon)} \frac{1}{(p^2)^{1/2+\epsilon}} \\ &\equiv T\Lambda^{2\epsilon} \frac{A_\epsilon}{8(p^2)^{1/2+\epsilon}} = \frac{T}{8p} + \mathcal{O}(\epsilon), \end{aligned} \quad (\text{A.55})$$

which leads to an IR singularity in the integral defining $S_2^{\text{I,b}}$ but cannot be separated from it without causing new UV divergences.

To facilitate an analytic evaluation of the divergence, we add and subtract from the integrand of $S_2^{\text{I,b}}$ a term of the form $\left[\Pi_{\text{IR}}^{(T)}(p_0 = 0, p)\right]^3 \frac{m^2}{p^2 + m^2}$, where the in principle arbitrary regularization mass m is chosen to be $m = 2\pi T$ for computational convenience. This enables us to write

$$\begin{aligned} S_2^{\text{I,b}} &= T \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left\{ \left[\Pi^{(T)}(p_0 = 0, p)\right]^3 - \left(\frac{T}{8p}\right)^3 \frac{(2\pi T)^2}{p^2 + (2\pi T)^2} \right\} + \\ &\quad + T\Lambda^{2\epsilon} \int \frac{d^{3-2\epsilon}\mathbf{p}}{(2\pi)^{3-2\epsilon}} \left[\Pi_{\text{IR}}^{(T)}(p_0 = 0, p)\right]^3 \frac{(2\pi T)^2}{p^2 + (2\pi T)^2} + \mathcal{O}(\epsilon), \end{aligned} \quad (\text{A.56})$$

where the first piece has the virtue of being finite at $d = 3$, while the second one is straightforward to evaluate analytically. Substituting here the Fourier representation of $\Pi^{(T)}(P)$ as well as the above form of $\Pi_{\text{IR}}^{(T)}(p_0 = 0, p)$, we obtain after scaling all variables dimensionless

$$\begin{aligned} S_2^{\text{I,b}} &= \frac{T^4}{64(4\pi)^2} \int_0^\infty dp \frac{1}{p} \left\{ \left[\frac{2}{\pi} \int_0^\infty dr \frac{\sin(pr)}{r} \left(\coth r - \frac{1}{r} \right) \right]^3 - \frac{1}{p^2 + 1} \right\} + \\ &\quad + \frac{T^4 \Lambda^{8\epsilon} A_\epsilon^3}{512} \frac{(2\pi T)^{-8\epsilon}}{(2\pi)^{3-2\epsilon}} \frac{2\pi^{3/2-\epsilon}}{\Gamma(3/2 - \epsilon)} \int_0^\infty dp \frac{p^{-1-8\epsilon}}{p^2 + 1} + \mathcal{O}(\epsilon) \\ &= -\frac{T^4}{512(4\pi)^2} \left\{ \frac{1}{\epsilon} + 4 \ln \frac{\Lambda^2}{4\pi T^2} + 2 + 12 \ln 2 - 4\gamma_E + 20.91614600219(1) + \mathcal{O}(\epsilon) \right\} \\ &= -\frac{T^4}{512(4\pi)^2} \left\{ \frac{1}{\epsilon} + 4 \ln \frac{\Lambda^2}{4\pi T^2} + 28.92504950930(1) + \mathcal{O}(\epsilon) \right\}. \end{aligned} \quad (\text{A.57})$$

A.3.5 $S_2^{\text{II,b}}$, $S_2^{\text{III,b}}$, $S_2^{\text{III,c}}$, S_2^{IV} and S_2^{V}

The remaining five sum-integrals are again obtained through a straightforward application of eq. (2.19), as they can be written in the forms

$$S_2^{\text{II,b}} = 12 (\mathcal{I}_1)^2 \left\{ \beta \mathcal{I}_{2+\epsilon} - \frac{1}{(4\pi)^{2\epsilon}} \mathcal{I}_2 \right\}, \quad (\text{A.58})$$

$$S_2^{\text{III,b}} = 6\mathcal{I}_1 \left\{ \beta^2 \mathcal{I}_{1+2\epsilon} - \frac{2\beta}{(4\pi)^2 \epsilon} \mathcal{I}_{1+\epsilon} + \frac{1}{(4\pi)^4 \epsilon^2} \mathcal{I}_1 \right\}, \quad (\text{A.59})$$

$$S_2^{\text{III,c}} = \frac{24J_1 T^4}{3-2\epsilon} \left\{ \beta^2 \mathcal{I}_{2+2\epsilon} - \frac{2\beta}{(4\pi)^2 \epsilon} \mathcal{I}_{2+\epsilon} + \frac{1}{(4\pi)^4 \epsilon^2} \mathcal{I}_2 - (4-2\epsilon) \left[\beta^2 \mathcal{I}_{3+2\epsilon}^2 - \frac{2\beta}{(4\pi)^2 \epsilon} \mathcal{I}_{3+\epsilon}^2 + \frac{1}{(4\pi)^4 \epsilon^2} \mathcal{I}_3^2 \right] \right\}, \quad (\text{A.60})$$

$$S_2^{\text{IV}} = \beta^3 \mathcal{I}_{3\epsilon} - \frac{3\beta^2}{(4\pi)^2 \epsilon} \mathcal{I}_{2\epsilon} + \frac{3\beta}{(4\pi)^4 \epsilon^2} \mathcal{I}_\epsilon, \quad (\text{A.61})$$

$$S_2^{\text{V}} = -\frac{3}{(4\pi)^4 \epsilon^2} (\mathcal{I}_1)^2, \quad (\text{A.62})$$

the sum of which gives

$$\begin{aligned} & S_2^{\text{II,b}} + S_2^{\text{III,b}} + S_2^{\text{III,c}} + S_2^{\text{IV}} + S_2^{\text{V}} \\ &= -\frac{T^4}{16(4\pi)^4} \left\{ \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \left[2 \ln \frac{\Lambda^2}{4\pi T^2} + \frac{10}{3} - 2\gamma_E + 4 \frac{\zeta'(-1)}{\zeta(-1)} \right] + \left(\ln \frac{\Lambda^2}{4\pi T^2} \right)^2 - \right. \\ & \quad - \left[\frac{2}{9} + \frac{46\gamma_E}{15} - \frac{16}{3} \frac{\zeta'(-1)}{\zeta(-1)} + \frac{4}{15} \frac{\zeta'(-3)}{\zeta(-3)} \right] \ln \frac{\Lambda^2}{4\pi T^2} - \frac{193}{27} + \frac{\pi^2}{6} - \frac{74\gamma_E}{45} + \\ & \quad + \frac{31\gamma_E^2}{15} + \frac{8(1-2\gamma_E)}{3} \frac{\zeta'(-1)}{\zeta(-1)} + 4 \left(\frac{\zeta'(-1)}{\zeta(-1)} \right)^2 - \frac{4(14-3\gamma_E)}{45} \frac{\zeta'(-3)}{\zeta(-3)} + \\ & \quad \left. + \frac{4}{3} \frac{\zeta''(-1)}{\zeta(-1)} - \frac{4}{15} \frac{\zeta''(-3)}{\zeta(-3)} + \frac{32}{15} \gamma_1 \right\} + \mathcal{O}(\epsilon). \end{aligned} \quad (\text{A.63})$$

A.3.6 The full result for S_2

Taking the sum of all the various pieces of S_2 displayed above, we can now finally write down the result for the entire sum-integral in the form

$$\begin{aligned} S_2 &= -\frac{T^4}{16(4\pi)^4} \left\{ \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \left[2 \ln \frac{\Lambda^2}{4\pi T^2} + \frac{10}{3} - 2\gamma_E + 4 \frac{\zeta'(-1)}{\zeta(-1)} \right] + \left(\ln \frac{\Lambda^2}{4\pi T^2} \right)^2 + \right. \\ & \quad + \left[\frac{6}{5} - 2\gamma_E + 4 \frac{\zeta'(-3)}{\zeta(-3)} \right] \ln \frac{\Lambda^2}{4\pi T^2} - \frac{581}{135} + \frac{\pi^2}{6} + \frac{32(4+3\gamma_E)}{45} \ln 2 - \frac{14\gamma_E}{15} + \gamma_E^2 - \\ & \quad - \frac{8(3+4 \ln 2)}{3} \frac{\zeta'(-1)}{\zeta(-1)} + 4 \left(\frac{\zeta'(-1)}{\zeta(-1)} \right)^2 + \left(\frac{328}{45} + \frac{128 \ln 2}{15} - 4\gamma_E \right) \frac{\zeta'(-3)}{\zeta(-3)} - \\ & \quad - \frac{96}{\pi^2} [(1-\gamma_E - \ln \pi)\zeta(3) + \zeta'(3)] + \frac{4}{3} \frac{\zeta''(-1)}{\zeta(-1)} - \frac{4}{15} \frac{\zeta''(-3)}{\zeta(-3)} + \\ & \quad \left. + \frac{32}{15} \gamma_1 - 32.575396998(2) \right\} - \\ & \quad - \frac{T^4}{512(4\pi)^2} \left\{ \frac{1}{\epsilon} + 4 \ln \frac{\Lambda^2}{4\pi T^2} + 2 + 12 \ln 2 - 4\gamma_E + 20.91614600219(1) \right\} + \mathcal{O}(\epsilon). \end{aligned} \quad (\text{A.64})$$

On the last row, we have separated the IR-singular contribution from $S_2^{\text{I,b}}$ in eq. (A.57). Evaluating the non-logarithmic terms numerically leads to the result in eq. (2.24).

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